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Quantum field theory on Clifford–Klein space–times. The effective Lagrangian and vacuum stress–energy tensor

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Abstract. The vacuum-averaged stress–energy tensor is calculated for a scalar field on a variety of static space–times, $T \otimes M_3$, where the spatial section, M_3, is a Clifford–Klein space form of the flat or spherical type, R^3/Γ, or S^3/Γ. The particular examples when M_3 is a Klein-bottle waveguide, $R \otimes K_2$, or a lens-space, S^3/Z_m, are treated in most detail. It is found that the vacuum stress on quotient spaces is not of the same tensorial structure as $R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. This leads to difficulties with the back-reaction problem. It is further found that 'twisting' the field alters the vacuum stress compared to the untwisted theory. Values for the total energies in the three types of polyhedral cellular decompositions of S^3 are given. Dirichlet boundary conditions for rectangular cavities are also considered.

1. Introduction

In this paper we wish to present some extensions and applications of a zeta-function technique introduced earlier (Dowker and Critchley 1976a, 1977, Dowker and Kennedy 1978, Hawking 1977) in connection with external field problems, in particular with quantum field theory in curved spaces.

The interest lies in considering spaces with boundaries and topologically non-trivial space–times, especially those that are multiply connected. Our discussion and approach are different to those of Isham (1978) whose interesting and related work was received whilst ours was in progress. For convenience we have adopted his terminology of 'twisted' fields.

In the next section some general theory is given which is then applied to different systems chosen to bring out various aspects of the formalism. The Klein-bottle waveguide is discussed in §§ 4, 7 while § 9 is concerned with curved space examples—manifolds locally isometric spatially to the Einstein universe.

The quantity we shall be concentrating on will be the vacuum-averaged stress–energy tensor of a scalar field. This is only one aspect of the quantum field theory, of course, and a discussion of the Fock space constructions is deferred to another time.

2. Quantum field theory on multiply connected spaces

The basic formalism has been given elsewhere (Dowker 1972a, b) but will be recapitulated here. Let M be a multiply connected Riemannian space of Euclidean signature. It could be, for example, the Euclidean section of a space–time or it could be the

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spatial section of a static manifold. M is given as the quotient space

$$M = \tilde{M}/\Gamma \tag{1}$$

where $\tilde{M}$ is the universal covering space of M and Γ is isomorphic to the fundamental group of M, $\pi_1(M)$. Γ is a symmetry group of M, is discontinuous, acts freely and without fixed points (e.g. Wolf 1967).

Quantum mechanics on M has been discussed by Laidlaw and DeWitt (1971), Schulman (1971), Dowker (1972a) and Finkelstein and Rubenstein (1968). We shall make use of the descriptions and notations of Dowker (1972a).

For simplicity, in this paper, we shall deal with functions on M, as in scalar field theory, and begin by writing down the expression for the kernel K of the heat equation on M, in terms of $\tilde{K}$, that on $\tilde{M}$,

$$K(q_0', q_0'', \tau) = \sum_\gamma a(\gamma)\tilde{K}(q_0', q_0''\gamma, \tau). \tag{2}$$

The connection with quantum field theory comes about when we identify $-i\tau$ with the proper time in the Fock–Schwinger–DeWitt fifth-parameter formalism (Fock 1937, Schwinger 1951, DeWitt 1975).

On the left of equation (2), q_0' and q_0'' refer to points of M while on the right they stand for the two fixed pre-images in $\tilde{M}$ of these points. The set of these pre-images is isomorphic to M and is such that all points of $\tilde{M}$ can be generated from it by application of Γ, $q_0\gamma$. The multiplier, $a(\gamma)$, is a unitary, one-dimensional representation of Γ, i.e.

$$a(\gamma_1)a(\gamma_2) = a(\gamma_2\gamma_1). \tag{3}$$

$\tilde{K}$ satisfies an equation

$$\frac{\partial}{\partial\tau}\tilde{K} + \tilde{A}\tilde{K} = \tilde{1}$$

and K

$$\frac{\partial}{\partial\tau}K + AK = 1$$

where A is the restriction of $\tilde{A}$ to M. For A we have in mind a Laplace operator.

In terms of the eigenproblem

$$\tilde{A}|\tilde{n}\rangle = \tilde{\lambda}_n|\tilde{n}\rangle$$

$\tilde{K}$ can be formally written (we assume $\tilde{\lambda}_n \geqslant 0$):

$$\tilde{K} = e^{-\tilde{A}\tau} = \sum_n e^{-\tilde{\lambda}_n\tau}|\tilde{n}\rangle\langle\tilde{n}|. \tag{4}$$

A function on M can be thought of as a function on $\tilde{M}$, $\tilde{\Psi}$, which satisfies

$$\tilde{\Psi}(q_0\gamma) = a(\gamma)\tilde{\Psi}(q_0)$$

or, in Dirac notation,

$$\langle q_0\gamma|\tilde{\Psi}\rangle = a(\gamma)\langle q_0|\tilde{\Psi}\rangle.$$

Define the operator $\hat{\gamma}$ by

$$\langle q_0\gamma|\tilde{\psi}\rangle = \langle q_0|\hat{\gamma}^{-1}|\tilde{\psi}\rangle \tag{5}$$

and assume that it commutes with $\tilde{A}$ which it will if $\tilde{A}$ is a covariant operator on $\tilde{M}$ since Γ is a symmetry group of $\tilde{M}$. This means that, for example,

$$\tilde{K}(\tilde{q}', \tilde{q}''\gamma, \tau) = \tilde{K}(\tilde{q}'\gamma^{-1}, \tilde{q}'', \tau)$$

or

$$|\Gamma|^{-1} \sum_{\mu} \tilde{K}(\tilde{q}'\mu, \tilde{q}''\mu, \tau) = \tilde{K}(\tilde{q}', \tilde{q}'', \tau) \tag{6}$$

where the sum runs over all elements of Γ, of which there are $|\Gamma|$.

An equation like (6) will hold for matrix elements of all covariant (and therefore invariant) operators on $\tilde{M}$.

Using (6) it can easily be shown that K satisfies the semigroup property on M if $\tilde{K}$ satisfies it on $\tilde{M}$.

In Dowker and Critchley (1976a) it was shown that the effective Lagrangian was related to the zeta function of the covariant Laplacian and we turn to the relation of the zeta functions on M and $\tilde{M}$. This easily follows from (2).

In terms of the Mellin transforms we have the connection between ζ and K

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}\tau\, \tau^{s-1} K(\tau) \tag{7}$$

and between $\tilde{\zeta}$ and $\tilde{K}$

$$\tilde{\zeta}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{d}\tau\, \tau^{s-1} \tilde{K}(\tau) \tag{8}$$

where we have assumed that none of the $\tilde{\lambda}_n$ are zero. (A zero eigenvalue can easily be allowed for.)

It is a straightforward consequence of (2), (7) and (8) that ζ and $\tilde{\zeta}$ are related by the image sum

$$\zeta(s, q_0' \,|\, q_0'') = \sum_{\gamma} a(\gamma) \tilde{\zeta}(s, q_0' \,|\, q_0''\gamma). \tag{9}$$

In terms of the eigenvalues $\tilde{\lambda}_n$ and eigenvectors $|\tilde{n}\rangle$ (9) can be rewritten

$$\zeta(s, q_0' \,|\, q_0'') = \sum_{\gamma, n} a(\gamma) \frac{\langle q_0' \,|\, \tilde{n}\rangle\langle \tilde{n} \,|\, q_0''\gamma\rangle}{\tilde{\lambda}_n^s}$$

or, from (6),

$$\zeta(s, q_0' \,|\, q_0'') = |\Gamma|^{-1} \sum_{\mu, \gamma, n} a(\gamma) \frac{\langle q_0'\mu \,|\, \tilde{n}\rangle\langle \tilde{n} \,|\, q_0''\gamma\mu\rangle}{\tilde{\lambda}_n^s}.$$

The sum over γ can be re-arranged by the replacement $\gamma \to \mu\gamma\mu^{-1}$ and if (5) is used there results

$$\zeta(s, q_0' \,|\, q_0'') = |\Gamma|^{-1} \sum_{\mu, \gamma, n} a(\gamma) \frac{\langle q_0'\mu \,|\, \tilde{n}\rangle\langle \tilde{n} \,|\, \hat{\gamma} \,|\, q_0''\mu\rangle}{\tilde{\lambda}_n^s}. \tag{10}$$

The reason for these manipulations is that we require, in field theory, the coincidence limit $q_0'' = q_0'$ and then the integrated zeta function,

$$\int_M \zeta(s, q_0' \,|\, q_0') \,\mathrm{d}q_0' \equiv \zeta(s).$$

The sum over μ allows us to employ the completeness relation in $\tilde{M}$,

$$\sum_\mu \int_M |q'_0\mu\rangle\langle q'_0\mu|\,\mathrm{d}q'_0 = \tilde{1}$$

to give

$$\zeta(s) = |\Gamma|^{-1} \sum_{\gamma,n} a(\gamma) \frac{\langle \tilde{n}|\hat{\gamma}|\tilde{n}\rangle}{\tilde{\lambda}_n^s}. \tag{11}$$

The analysis is now standard quantum mechanics. The eigenspace spanned by the eigenvectors $|\tilde{n}\rangle$ corresponding to the same eigenvalue $\tilde{\lambda}_n$ forms the carrier space for a representation of the symmetry group Γ. Hence we can write (11) as

$$\zeta(s) = |\Gamma|^{-1} \sum_{\gamma,\tilde{\lambda}} a(\gamma)\tilde{\lambda}^{-s}\chi_{\tilde{\lambda}}(\gamma) \tag{12}$$

where the character $\chi_{\tilde{\lambda}}(\gamma)$ is the trace of $\hat{\gamma}$ in the $\tilde{\lambda}$-eigenspace. $a(\gamma)$ is a character. Let us rename it $\chi_a(\gamma)$ for uniformity and then note that

$$n_{a\tilde{\lambda}} = |\Gamma|^{-1} \sum_\gamma \chi_a(\gamma)\chi_{\tilde{\lambda}}(\gamma)$$

is the number of times the irreducible representation 'a' occurs in the $\tilde{\lambda}$-representation, so that $\zeta_a(s)$ 'simplifies' to

$$\zeta_a(s) = \sum_{\tilde{\lambda}} n_{a\tilde{\lambda}}\tilde{\lambda}^{-s}. \tag{13}$$

If $\Gamma = \mathbf{1}$, so that $M = \tilde{M}$, $n_{a\tilde{\lambda}}$ is just the degeneracy of the $\tilde{\lambda}$-eigenvalue, as is correct.

The identity (6) can be applied directly to (9) to yield

$$\zeta_a(s) = |\Gamma|^{-1}\tilde{\zeta}(s) + \sum_{\gamma\neq\mathbf{1}} a(\gamma)\tilde{\zeta}(s,\gamma) \tag{14}$$

where the term $\gamma = \mathbf{1}$ has been separated off and we have chosen $a(\mathbf{1}) = 1$, as we may. $\tilde{\zeta}(s)$ is the integrated zeta function on $\tilde{M}$ and $\tilde{\zeta}(s,\gamma)$ is defined by

$$\tilde{\zeta}(s,\gamma) = |\Gamma|^{-1} \sum_\mu \int_M \tilde{\zeta}(q'_0\mu\,|\,q'_0\mu\gamma)\,\mathrm{d}q'_0 = |\Gamma|^{-1}\int_{\tilde{M}} \tilde{\zeta}(\tilde{q}'\,|\,\tilde{q}'\gamma)\,\mathrm{d}\tilde{q}'. \tag{15}$$

As a simple example of some of these formulae the reader may care to derive the zeta-function on a torus, say a circle, from the eigenvalues and functions on flat Euclidean space using both box and δ-function normalisation. The answer can, of course, be written down immediately, but the calculation is instructive.

At this point it might be useful to mention that typically one of the things we are interested in is the one-loop effective action $W^{(1)}$. In Dowker and Critchley (1976a) it is shown that this is given by

$$W^{(1)} = -\lim_{s\to1} \frac{\mathrm{i}}{2}\left(\frac{\zeta(0)}{s-1} + \zeta'(0)\right),$$

with the divergences exhibited as a pole of residue $\zeta(0)$. From (14)

$$\zeta_a(0) = |\Gamma|^{-1}\tilde{\zeta}(0) + \sum_{\gamma\neq\mathbf{1}} a(\gamma)\tilde{\zeta}(0,\gamma)$$

but

$$\tilde{\zeta}(0,\gamma) = 0 \qquad \text{if } \gamma \neq \mathbf{1},$$

which follows from Minakshisundaram and Pleijel (1949, p 254) and so

$$\zeta_a(0) = |\Gamma|^{-1}\tilde{\zeta}(0).$$

This means that the divergences are the same in M as in $\tilde{M}$, allowing for the different volumes,

$$|\tilde{M}| = |\Gamma|\,|M|.$$

We require this to be so since the spaces are locally identical.

3. Simple examples

It can be seen from (13) that the effect of dividing $\tilde{M}$ by Γ is to select and re-arrange the eigenvalues. If one wants to write down an explicit formula for $\zeta_a(s)$ in simple cases it is easiest to determine the eigenvalues and degeneracies appropriate to the periodicity condition,

$$\tilde{\psi}(q_0\gamma) = a(\gamma)\tilde{\psi}(q_0), \tag{17}$$

directly. For example if M is the one-torus, or circle, S^1 (θ: $0 \leqslant \theta \leqslant 2\pi$), $\tilde{M}$ is the real line R^1 and Γ is the infinite cyclic group Z_∞ such that each element is labelled by an integer, n. The effect of acting on θ by γ_n is a (lattice) translation $\theta\gamma_n = \theta + 2\pi n$. Then (17) becomes

$$\tilde{\psi}(\theta + 2\pi n) = a(\gamma_n)\tilde{\psi}(\theta) \tag{18}$$

with in general

$$a(\gamma_n) = \exp(2\pi i n\alpha), \qquad 0 \leqslant \alpha \leqslant \tfrac{1}{2}, \tag{19}$$

as the one-dimensional representation of Z_∞, labelled by α. This is an oft-discussed example (e.g. Schulman 1968, 1971, Dowker 1977).

The eigenvalues and functions of the Laplacian $\mathrm{d}^2/\mathrm{d}\theta^2$ are $-k^2$ and $\exp(\mathrm{i}k\theta)/(2\pi)^{1/2}$ where, from (18) and (19), $k = n - \alpha$, with n an integer. The fields on S^1 can thus be labelled by the parameter α, say $\phi^{(\alpha)}$.

As a quantity of physical interest we can calculate the vacuum average of the Hamiltonian $E = \langle \hat{H} \rangle$ for the $\phi^{(\alpha)}$ fields. It is shown in Dowker and Kennedy (1978) that in flat space E is given by

$$E = \tfrac{1}{2}\mathrm{i}\,\mathrm{tr}_{d-1}[\boldsymbol{\zeta}'_d(0)] \times 2 \tag{20}$$

where $[\boldsymbol{\zeta}_d(s)]$ is the equal-time zeta-function on d-dimensional space–time, and tr_{d-1} stands for an integration over the spatial coordinates. The extra factor of 2 occurs because we are considering complex fields. $[\boldsymbol{\zeta}_d(s)]$ can be related to $\boldsymbol{\zeta}_{d-1}(s)$, the zeta function on the spatial section, by

$$[\boldsymbol{\zeta}_d(s)] = \mathrm{i}(4\pi)^{-1/2}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\boldsymbol{\zeta}_{d-1}(s-\tfrac{1}{2}), \tag{21}$$

so that for E we have, therefore,

$$E = \tfrac{1}{2}\zeta_{d-1}(-\tfrac{1}{2}) \times 2 \tag{22}$$

where $\zeta_{d-1}(s)$ is the *integrated* zeta function on the spatial section,

$$\zeta_{d-1}s) \equiv \mathrm{tr}_{d-1\ d-1}(s).$$

If $d=2$ we have for the $\phi^{(\alpha)}$ fields introduced above

$$\zeta_1^{(\alpha)}(s)=\sum_k k^{-2s}=\sum_{n=-\infty}^{\infty}(n-\alpha)^{-2s}.$$

If α is zero the zero-eigenvalue, $n=\alpha$, is omitted from the sum. Re-arranging the terms leads to the expression

$$\zeta_1^{(\alpha)}(s)=\zeta_{\rm R}(2s,\alpha)+\zeta_{\rm R}(2s,1-\alpha)$$

in terms of the Hurwitz–Lerch–Hermite zeta function (Whittaker and Watson 1915),

$$\zeta_{\rm R}(s,w)=\sum_{n=0}^{\infty}(n+w)^{-s},$$

which has the specific values

$$\zeta_{\rm R}(-p,w)=-\frac{1}{p+1}\varphi_{p+1}(w)$$

where the $\varphi_p(w)$ are related to the Bernoulli polynomials (Lindelöf 1905), and have the property $\varphi_{2k}(1-w)=\varphi_{2k}(w)$. Then for E we find

$$E=-\tfrac{1}{2}\phi_2(\alpha)\times 2=-\pi^{-2}\sum_{n=1}^{\infty}n^{-2}\cos(2n\pi\alpha)=\alpha-\alpha^2-\tfrac{1}{6}.$$

Two special cases are $\alpha=0$, for which

$$E=E^{(0)}=-\tfrac{1}{12}\times 2 \tag{23}$$

and $\alpha=\frac{1}{2}$, for which

$$E=E^{(1/2)}=+\tfrac{1}{24}\times 2. \tag{24}$$

The value (23) agrees with that first calculated by Ford (1975) who considered real fields, so no factor of two, and used an energy cut-off. The value (24) is the same, apart again from the factor of two due to the complexification, as the one evaluated by Isham (1978), for what he terms 'twisted' (real) scalar fields, using an 'energy zeta-function' method (cf Gibbons 1977). It can be seen from (18) and (19) that for real fields, $\phi^{(\alpha)}$, the only possible values of α are 0 and $\frac{1}{2}$.

$\alpha=0$ corresponds to the trivial, $a(\gamma_n)=1$, representation of Z_∞ while $\alpha=\frac{1}{2}$ gives the non-trivial representation factored by Z_2, $a(\gamma_n)=(-1)^n$.

In general for real fields $a(\gamma)$ must be real also, which means that every $a(\gamma)$ is either plus or minus one. This gives our classification of these fields on M, which is precisely the same as Isham's (1978) expressed in terms of the homomorphisms of $\pi_1(M)$ into Z_2 ($\sim H^1(\pi_1;Z_2)\sim H^1(M;Z_2)$).

Isham proves that there are no complex twisted scalar fields for $M=S^1$. Our development seems to lack such a restriction in that the fields $\phi^{(\alpha)}$ seem to interpolate quite nicely between the two limits $\alpha=0$ and $\alpha=\frac{1}{2}$. It is possible that these fields do not qualify as 'twisted' on Isham's definition.

The significance of the parameter α in the case of *quantum mechanics* on multiply connected spaces has been discussed by Schulman (1971). In general terms, on the multiply connected space S^1 the Hamiltonian is not essentially self-adjoint and admits a one-parameter family of extensions each member of which describes a different physical situation. One precise physical possibility is that α is proportional to the flux in an impenetrable solenoid passing through S^1, as in the Aharonov–Bohm effect. In this example we see that α is not a geometric quantity intrinsic to the manifold.

In general there are two basic methods available for calculating quantities of interest such as ζ. One is that employed above and could be called the eigenvalue approach. The other is to obtain ζ from $\tilde{G}$ by a non-eigenvalue method, such as images in flat space. To illustrate these two methods we shall consider the case when M is three dimensional and is a Klein bottle, K_2, in the x–y plane, times the z axis, Z. We are interested in finding the energy density $\langle \hat{T}_{00}\rangle$ and the total energy per unit z slice of this 'waveguide'. The ordinary rectangular waveguide was treated by Dowker and Kennedy (1978) and some comments can be found below in § 5.

4. The Klein-bottle waveguide, $\mathscr{K}_2$

Firstly we will use the direct image method (Dowker and Critchley 1976b, Brown and Maclay 1969) which actually does not involve zeta-function regularisation, the point being that it is only the $\gamma = \mathbf{1}$ term in (9) that produces the divergences and, in flat space–time, the renormalisation (or subtraction) procedure is just to drop this term since it corresponds to the infinite free space contribution (cf also Candelas and Deutsch 1977). There is then no need to introduce ζ, and $\langle \hat{T}_{00}\rangle$ is evaluated as a coincidence limit of a differential operator acting on the Feynman Green function $G(x, x') = \zeta(1, x, x')$ in a, by now, standard fashion.

For the conformally coupled real scalar field in flat space–time we have the particular expression

$$\langle \hat{T}_{00}(x)\rangle = \lim_{x'\to x} \mathrm{i}[(2\xi + \tfrac{1}{2})\,\partial_0\,\partial_{0'} + (2\xi - \tfrac{1}{2})\,\partial_i\,\partial_{i'}]G(x, x') \equiv [\vec{T}_{00}G] \tag{25}$$

with $\xi = \frac{1}{6}$.

The Klein bottle can be regarded (e.g. Wolf 1967, Kobayashi and Nomizu 1963) as the real plane, R^2, factored by the group that identifies (x, y) with $(x + pa, (-1)^p y + 2mb)$, for integer p and m. The actual Klein bottle K_2 can be taken as the region $0 \leqslant x \leqslant a$, $-b \leqslant y \leqslant b$ so that its area is $2ab$. The region $0 \leqslant x \leqslant 2a$, $-b \leqslant y \leqslant b$ provides a double covering of K_2.

The image form of $G(x, x')$, (9), can now be written down by hand,

$$G(x, x') = \frac{\mathrm{i}}{4\pi^2} \sum_{n,m=-\infty}^{\infty} [\![[(x - x' - 2na)^2 + (y - y' - 2mb)^2 + (z - z')^2 - (t - t')^2]^{-1}$$
$$+ \{[x - x' - (2n+1)a]^2 + (y + y' - 2mb)^2 + (z - z')^2 - (t - t')^2\}^{-1}]\!]. \tag{26}$$

For simplicity we have taken all the $a(\gamma)$ equal to unity, that is, the boundary conditions are the standard 'periodic' ones. Other possibilities will be treated later in § 7.

The first sum in (26) is just the Green function for an ordinary rectangle with sides $2a$ and $2b$. The corresponding constant energy density is that derived by Dowker and Critchley (1976b),

$$\langle \hat{T}_{00}\rangle_1 = -(32\pi^2)^{-1} \sum_{-\infty}^{\infty}{}' (n^2a^2 + m^2b^2)^{-2} \tag{27a}$$

(a result independent of ξ).

By contrast, the second sum in (26), produces an energy density which is a function of y,

$$\langle\hat{T}_{00}\rangle_2=\pi^{-2}\sum_{-\infty}^{\infty}\left(\frac{2a^2(4\xi-1)(2n+1)^2}{[(2n+1)^2a^2+4(y-mb)^2]^3}-\frac{6\xi-1}{[(2n+1)^2a^2+4(y-mb)^2]^2}\right). \tag{27b}$$

For the conformally coupled case the complete energy density is

$$\langle\hat{T}_{00}\rangle_{\mathscr{K}_2}=-(32\pi^2)^{-1}\sum_{-\infty}^{\infty}{}'(n^2a^2+m^2b^2)^{-2}-\frac{2a^2}{3\pi^2}\sum_{-\infty}^{\infty}\frac{(2n+1)^2}{[(2n+1)^2a^2+4(y-mb)^2]^3}. \tag{28}$$

The infinite Möbius strip value is obtained by letting b tend to infinity. This picks out the $m=0$ term in the double sum to give

$$\langle\hat{T}_{00}\rangle_{\mathscr{M}_2}=-(16\pi^2a^4)^{-1}\zeta_R(4)-\frac{2a^2}{3\pi^2}\sum_{-\infty}^{\infty}\frac{(2n+1)^2}{[(2n+1)^2a^2+4y^2]^3} \tag{29}$$

which can be evaluated to yield

$$\langle\hat{T}_{00}\rangle_{\mathscr{M}_2}=\frac{\pi^2}{192a^4}[(\eta^{-2}-2\eta^{-1}\tanh\eta)\operatorname{sech}^2\eta-\eta^{-3}\tanh\eta-\tfrac{2}{15}]$$

where $\eta=\pi y/a$. For small η it is found that

$$\langle\hat{T}_{00}\rangle_{\mathscr{M}_2}\sim\frac{\pi^2}{480a^4}(-7+8\eta^2)$$

while for large η

$$\langle\hat{T}_{00}\rangle_{\mathscr{M}_2}\sim-\frac{\pi^2}{192a^4}(\tfrac{2}{15}+\eta^{-3}).$$

A plot of this distribution is given in figure 1.

We note that $\langle\hat{T}_{00}(y)\rangle_{\mathscr{K}_2}$ is periodic in y with period b. It is also even in y and symmetrical about $y=b/2$, for positive y. A straightforward integration over the Klein bottle produces the total energy for a unit distance in the z direction $E(a, b)$ as

$$E(a,b)=-7(16\pi a^2)^{-1}\zeta_R(3)-ab(16\pi^2)^{-1}\sum_{n,m=-\infty}^{\infty}{}'(n^2a^2+m^2b^2)^{-2} \tag{30}$$

an expression valid for all ξ. (The reason for this is that the ξ-dependent terms are surface terms and go out on integration over $\mathscr{K}_2$ because $\partial\mathscr{K}_2=\varnothing$.)

The first part of (30) comes from the y-dependent term. The second part is the rectangle expression, and is just an Epstein zeta function (Epstein 1903, 1907) as indeed so are the terms in the density (28), or (29).

Since the properties of this zeta function are very relevant to our calculation they will be given here. Firstly define the two-dimensional Epstein zeta function Z by

$$Z\left|\begin{smallmatrix}g\\h\end{smallmatrix}\right|(s,A)=\sum_{m_1,m_2=-\infty}^{\infty}{}'\exp(2\pi i\tilde{h}m)[(\widetilde{m+g})A(m+g)]^{-s}\qquad \operatorname{Re}s>1 \tag{31}$$

where m, g, and h are column matrices, $m=\binom{m_1}{m_2}$, etc. A is a 2×2 matrix and the tilde indicates transpose.

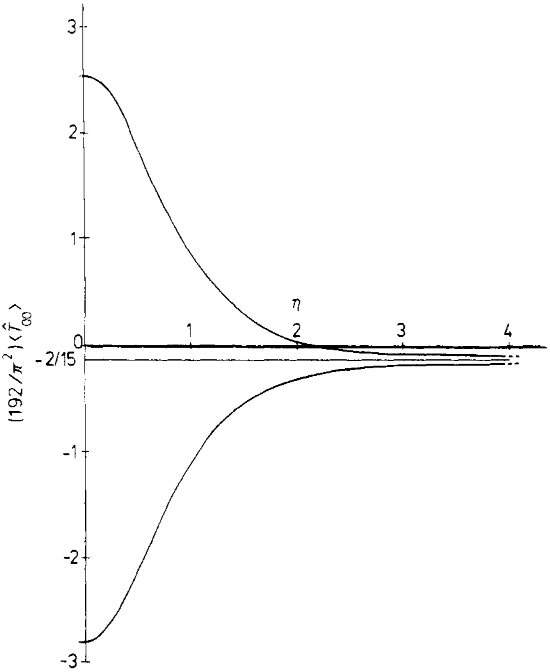

Figure 1. Vacuum energy density on an infinite Möbius strip, of unit circumference, as a function of the distance y from the centre of the band ($\eta = \pi y$). The lower curve is for a normal field and the upper for a twisted one ($\alpha = \frac{1}{2}$). The curves are symmetrical about $\eta = 0$.

Z satisfies the functional relation

$$Z\left|{}^{\tilde{g}}_{\tilde{h}}\right|(s, A) = (\det A)^{-1/2}\pi^{2s-1}\frac{\Gamma(1-s)}{\Gamma(s)}\exp(-2\pi i\tilde{g}h)Z\left|{}^{\tilde{h}}_{-\tilde{g}}\right|(1-s, A^{-1}) \tag{32}$$

which is, for us, the basic equation and links the image and eigenfunction approaches, as will be seen. In all our applications with rectangular regions the matrix A is diagonal. Equation (32) is essentially just the transformation equation for multiple theta functions used previously (Dowker 1971) to show the equivalence of eigenfunction and classical path, or image, methods.

The previous expressions will now be rewritten in terms of Z. Firstly the density $\langle\hat{T}_{00}\rangle_1 + \langle\hat{T}_{00}\rangle_2$ from (27):

$$\langle\hat{T}_{00}\rangle_{\mathcal{K}_2} = -(32\pi^2)^{-1}Z\left|{}^{0\ 0}_{0\ 0}\right|(2, A) - (16\pi^2)^{-1}\left((4\xi-1)a^2\frac{\partial}{\partial a^2} + 6\xi - 1\right)Z\left|{}^{\frac{1}{2}\ y/b}_{0\ \ 0}\right|(2, A) \tag{33}$$

where

$$A = \begin{Vmatrix} a^2 & 0 \\ 0 & b^2 \end{Vmatrix}.$$

Next, for the total energy $E(a, b)$ we find from (30)

$$E(a, b) = -(7/16\pi a^2)\zeta_{\mathrm{R}}(3) - (ab/16\pi^2)Z\left|{}^{0\ 0}_{0\ 0}\right|(2, A). \tag{34}$$

For the special case $a = b$ the Epstein zeta function is given by a sum of Hardy's

$$Z\begin{vmatrix}0&0\\0&0\end{vmatrix}(s, 1) = 4\zeta_{\mathrm{R}}(s)\beta(s)$$

and we have

$$E(a, a) = -(1/4\pi^2 a^2)[(7\pi/4)\zeta_{\mathrm{R}}(3) - \zeta_{\mathrm{R}}(2)\beta(2)]. \tag{35}$$

The function defined by (31) and its analytic continuation is an entire function of s unless the components of h are all integers, when $Z(s)$ has a pole at $s = 1$ with residue equal to $\pi(\det A)^{-1/2}$. $Z(s)$ vanishes when s is a negative integer and also when s is zero, unless the components of g are all integers when its value is $-\exp(-2\pi i \tilde{g} h)$.

After this slight digression we turn to the eigenvalue method and will firstly construct the zeta function for the Klein bottle (with periodic boundary conditions). The eigenproblem is discussed by Berger *et al* (1971) and we find for the 'non-local' zeta function

$$\begin{aligned}\zeta_{K_2}(s, x, y\,|\,x', y') &= (ab)^{-1}\Bigg(\sum_{\substack{m=-\infty\\(\mathrm{even})}}^{\infty}{}' \sum_{n=0}^{\infty} \epsilon_n \frac{\mathrm{e}^{\pi i m(x-x')/a} \cos(ny\pi/b)\cos(ny'\pi/b)}{\lambda_{mn}^{s}} \\ &\quad + \sum_{\substack{m=-\infty\\(\mathrm{odd})}}^{\infty} \sum_{n=1}^{\infty} \frac{\mathrm{e}^{\pi i m(x-x')/a} \sin(ny\pi/b)\sin(ny'\pi/b)}{\lambda_{mn}^{s}}\Bigg), \qquad \mathrm{Re}\, s > 1, \end{aligned} \tag{36}$$

where $\epsilon_0 = \frac{1}{2}$, and $\epsilon_n = 1$ for $n = 1, 2, \ldots$ and the eigenvalues λ_{mn} are given by

$$\lambda_{mn} = \pi^2(m^2 a^{-2} + n^2 b^{-2}).$$

(For those who wish to compare our expressions with the discussion in Berger *et al* (1971) note that our a and b are half those of Berger *et al* and that there is a factor of 2π missing in the y-modes of this reference.)

In the following calculation various coincidence limits of ζ_{K_2} and its derivatives will be needed. We give them now for convenience. Firstly ζ_{K_2} itself, which, after some slight manipulation is

$$\begin{aligned}\zeta_{K_2}(s, x, y\,|\,x, y) &= (2ab)^{-1}\Bigg(\sum_{m=1}^{\infty} (2\lambda_{2m,0}^{-s} - \lambda_{m,0}^{-s}) + \tfrac{1}{2}\sum_{-\infty}^{\infty} \lambda_{mn}^{-s} \\ &\quad + \sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty} (-1)^m \lambda_{mn}^{-s} \cos(2n\pi y/b)\Bigg). \end{aligned} \tag{37}$$

This expression can be more simply rewritten in terms of the Epstein zeta function,

$$\zeta_{K_2}(s, x, y\,|\,x, y) = (4ab\pi^s)^{-1}\left[Z\begin{vmatrix}0&0\\0&0\end{vmatrix}(s, A^{-1}) + Z\begin{vmatrix}0&0\\\frac{1}{2}&y/b\end{vmatrix}(s, A^{-1})\right] \tag{38}$$

a result which follows more directly by writing (36) in terms of Z,

$$\zeta_{K_2}(s, x, y\,|\,x', y') = (4ab\pi^s)^{-1}\left(Z\begin{vmatrix}0 & 0\\ \dfrac{x-x'}{2a} & \dfrac{y-y'}{2b}\end{vmatrix}(s, A^{-1}) + Z\begin{vmatrix}0 & 0\\ \dfrac{x-x'}{2a}+\dfrac{1}{2} & \dfrac{y+y'}{2b}\end{vmatrix}(s, A^{-1})\right). \tag{39}$$

We now wish to find expressions for the coincidence limits of derivatives of ζ_{K_2}. For this it will be convenient to make use of the relation

$$\frac{\partial^2}{\partial h_i\,\partial h_j} Z\left|{}^{0}_{h}\right|(s, A) = \frac{4\pi^2}{s-1}\frac{\partial}{\partial A^{ij}} Z\left|{}^{0}_{h}\right|(s-1, A) \tag{40}$$

which follows from the definition (31). This is useful because we have for (39) that

$$\begin{aligned}\frac{\partial}{\partial x}\frac{\partial}{\partial x'} &\equiv -(4a^2)^{-1}\frac{\partial}{\partial h_1}\frac{\partial}{\partial h_1}\\ \frac{\partial}{\partial y}\frac{\partial}{\partial y'} &\equiv -(4b^2)^{-1}\frac{\partial}{\partial h_2}\frac{\partial}{\partial h_2} \qquad \text{for the first term}\end{aligned} \tag{41}$$

and

$$\frac{\partial}{\partial y}\frac{\partial}{\partial y'} \equiv (4b^2)^{-1}\frac{\partial}{\partial h_2}\frac{\partial}{\partial h_2} \qquad \text{for the second.}$$

We also have in general

$$(A)^{ij}\frac{\partial^2}{\partial h_i\,\partial h_j} Z\left|{}^{0}_{h}\right|(s, A) = -4\pi^2 Z\left|{}^{0}_{h}\right|(s-1, A). \tag{42}$$

Then we find

$$\begin{aligned}&\lim_{x'\to x}\left(\frac{\partial}{\partial x'}\frac{\partial}{\partial x}\zeta_{K_2}(s, x, y\,|\,x', y)\right)\\ &\qquad = \frac{\pi^{2-2s}a}{(s-1)4b}\frac{\partial}{\partial a^2}\left[Z\left|{}^{0\ 0}_{0\ 0}\right|(s-1, A^{-1}) + Z\left|{}^{0\ \ 0}_{\frac12\ y/b}\right|(s-1, A^{-1})\right]\end{aligned} \tag{43}$$

and

$$\begin{aligned}&\lim_{y'\to y}\left(\frac{\partial}{\partial y'}\frac{\partial}{\partial y}\zeta_{K_2}(s, x, y\,|\,x, y')\right)\\ &\qquad = \frac{\pi^{2-2s}}{4ab}\left(1 - \frac{a^2}{s-1}\frac{\partial}{\partial a^2}\right)\left[Z\left|{}^{0\ 0}_{0\ 0}\right|(s-1, A^{-1}) - Z\left|{}^{0\ \ 0}_{\frac12\ y/b}\right|(s-1, A^{-1})\right].\end{aligned} \tag{44}$$

As an application of these formulae the density (33) will be rederived. From (25) the zeta function expression for the density is obtained by replacing $G(x, x')$ by the zeta function,

$$\langle \hat{T}_{00}(x)\rangle = \lim_{s\to 1}[\vec{T}_{00}\zeta_4(s, x, x')] \tag{45}$$

where the subscript '4' indicates a space–time zeta function, $x = (t, x, y, z)$.

Our space–time is $T \otimes Z \otimes K_2$ so that the t and z dependences are essentially trivial and it is easily shown that

$$\lim_{x'\to x}\partial_0\,\partial_0\,\zeta_4(s, x, x') = \lim_{x'\to x}\partial_z\,\partial_{z'}\,\zeta_4(s, x, x') = \lim_{x'\to x}\frac{1}{2}\left(\frac{1}{s-1}\zeta_4(s-1, x, x')\right). \tag{46}$$

Further, the t and z coincidence limit of ζ_4 is given in terms of the zeta function in the

x–y plane (here the Klein-bottle zeta function) by (Dowker and Kennedy 1978)

$$\lim_{\substack{t'\to t \\ z'\to z}} \zeta_4(s, x, x') = \frac{\mathrm{i}}{4\pi}\frac{1}{s-1}\zeta_{K_2}(s-1, x, y \,|\, x', y'). \tag{47}$$

Remembering the form of $\vec{T}_{00}$ (equation (25)), the results (38), (43), (44), (46) and (47) can all be substituted into (45) to yield

$$\begin{aligned}\langle \hat{T}_{00}(y)\rangle = -\lim_{s\to 1}\frac{\pi^{3-2s}}{16ab(s-1)}\Big(&\frac{4\xi-1}{s-2}a^2\frac{\partial}{\partial a^2}Z\left|\begin{smallmatrix}0 & 0\\ \frac{1}{2} & y/b\end{smallmatrix}\right|(s-2, A^{-1})\\ &+\tfrac{1}{2}(4\xi-1)\Big[Z\left|\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right|(s-2, A^{-1}) - Z\left|\begin{smallmatrix}0 & 0\\ \frac{1}{2} & y/b\end{smallmatrix}\right|(s-2, A^{-1})\Big]\\ &+\frac{2\xi}{s-2}Z\left|\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right|(s-2, A^{-1})\Big).\end{aligned}$$

The right hand side is finite in the limit $s\to 1$, which is seen if the functional relation (32) is used. This leads, after a little algebra, precisely to expression (33) thus demonstrating the equivalence, for this problem, of the eigenvalue and image methods. A similar conclusion is also valid for all rectangular and cuboidal problems with periodic boundary conditions.

This result is hardly surprising in view of the basic equivalence of the two forms (25) and (45) with $G(x, x') = \zeta_4(1, x, x')$. The only difference is in the treatment of the infinities. In the zeta function eigenvalue method there happen, in the present case, to be no infinities. The residue of the pole at $s = 1$ is proportional to $\zeta_{K_2}(-1)$ which vanishes because $Z(s)$ is zero for s a negative integer. This is expected since space-time is flat and there are no boundaries.

In the image method the direct term, corresponding to the $m = 0 = n$ image, diverges in the coincidence limit and is simply removed, since it is the infinite free space (Minkowski) expression.

It is helpful just to check the relation $G(x, x') = \zeta_4(1, x, x')$. For simplicity, since the z and t dependences are not interesting, the z and t coincidence limits will be taken. From (39), (47) and (32) we find

$$\begin{aligned}\lim_{\substack{t'\to t \\ z'\to z}} \zeta_4(s, x, x') = \frac{\mathrm{i}}{16\pi^2}\frac{\Gamma(2-s)}{\Gamma(s)}\Bigg(&Z\begin{vmatrix}\dfrac{x-x'}{2a} & \dfrac{y-y'}{2b}\\ 0 & 0\end{vmatrix}(2-s, A)\\ &+Z\begin{vmatrix}\dfrac{x-x'}{2a}+\dfrac{1}{2} & \dfrac{y+y'}{2b}\\ 0 & 0\end{vmatrix}(2-s, A)\Bigg)\end{aligned}$$

corresponding precisely to the two terms in (26), as expected, when s is set equal to unity. The fact that (25) and (45) are equal is then almost guaranteed from the start, at least in this case.

The integrated zeta function on the Klein bottle is best found from (37) and is

$$\zeta_{K_2}(s) = \pi^{-2s}\zeta_{\mathrm{R}}(2s)a^{2s}(2^{1-2s}-1) + \tfrac{1}{2}\pi^{-2s}Z\left|\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right|(s, A^{-1}).$$

In terms of this the total energy E is given by (Dowker and Kennedy 1978),

$$E(a, b) = (8\pi)^{-1}\zeta'_{K_2}(-1)$$

which leads to

$$E(a, b) = (7\pi/4a^2)\zeta_R'(-1) + (\pi/16)Z'\Big|\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\Big|(-1, A)$$

and, again, use of the functional relations for the zeta functions reproduces exactly equation (34). This demonstration is, of course, only an integrated version of the local one preceding but is included because it is more rapid.

5. Dirichlet boundary conditions. The slab revisited

Some preliminary calculations and remarks have been made in our earlier work (Dowker and Kennedy 1978) concerning the more complicated cases of Dirichlet (D) and Neumann (N) boundary conditions. Consider the standard Casimir geometry of infinite parallel plates discussed in many places. We attack this firstly from the zeta-function eigenvalue direction. The eigenfunctions are, for D conditions,

$$(2/a)^{1/2}\sin(m\pi x/a), \qquad m = 1, 2, \ldots$$

and the zeta function is

$$\zeta_1(s, x\,|\,x') = \frac{2}{a}\sum_{m=1}^{\infty}\left(\frac{m\pi}{a}\right)^{-2s}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{m\pi x'}{a}\right) \tag{51}$$

which can be rewritten in terms of the one-dimensional Epstein zeta function Z, defined by

$$Z\Big|\begin{smallmatrix}g\\ h\end{smallmatrix}\Big|(s) = \sum_{m=-\infty}^{\infty}|m+g|^{-s}\exp(2\pi i m h), \qquad \operatorname{Re} s > 1, \tag{52}$$

(which is also related to the Hurwitz–Lerch zeta function). For ζ_1 we find

$$\zeta_1(s, x\,|\,x') = \tfrac{1}{2}\pi^{-2s}a^{2s-1}\Big[Z\Big|\begin{smallmatrix}0\\ (x-x')/2a\end{smallmatrix}\Big|(2s) - Z\Big|\begin{smallmatrix}0\\ (x+x')/2a\end{smallmatrix}\Big|(2s)\Big], \tag{53}$$

which can further be transformed into an image form using the relation

$$Z\Big|\begin{smallmatrix}g\\ h\end{smallmatrix}\Big|(2s) = \pi^{2s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}-s)}{\Gamma(s)}\,\mathrm{e}^{-2\pi i g h}Z\Big|\begin{smallmatrix}h\\ -g\end{smallmatrix}\Big|(1-2s). \tag{54}$$

Thus

$$\zeta_1(s, x\,|\,x') = \frac{a^{2s-1}}{2\pi^{1/2}}\frac{\Gamma(\frac{1}{2}-s)}{\Gamma(s)}\Big[Z\Big|\begin{smallmatrix}(x-x')/2a\\ 0\end{smallmatrix}\Big|(1-2s) - Z\Big|\begin{smallmatrix}(x+x')/2a\\ 0\end{smallmatrix}\Big|(1-2s)\Big]. \tag{55}$$

For $s = -\frac{1}{2}$ this yields the image form of the Green function and the transformation is very familiar in many topics, heat conduction being a good example (Hobson 1888, Sommerfeld 1949, Carslaw and Jaeger 1959).

In this case the t, z and y dependences are essentially trivial. For the t, z, y coincidence limit of the space–time zeta function we have

$$\lim_{\substack{t'\to t\\ z'\to z\\ y'\to y}}\zeta_4(s, x, x') = \frac{\mathrm{i}}{(4\pi)^{3/2}}\frac{\Gamma(s-\frac{3}{2})}{\Gamma(s)}\zeta_1(s-\tfrac{3}{2}, x\,|\,x'). \tag{56}$$

Further,

$$\lim_{x'\to x} \partial_0 \partial_0 \zeta_4(s, x, x')$$

$$= \lim_{x'\to x} \partial_z \partial_{z'} \zeta_4(s, x, x') = \lim_{x'\to x} \partial_y \partial_{y'} \zeta_4(s, x, x') = \frac{1}{2}\frac{1}{s-1}\zeta_4(s-1, x, x), \quad (57)$$

so that we can construct the energy density (45) in this reduced geometry.

From (51), or (53), the coincidence limits,

$$\zeta_1(s, x\,|\,x) = \pi^{-2s}a^{2s-1}\zeta_R(2s) - \tfrac{1}{2}\pi^{-2s}a^{2s-1}Z\left|{}^{0}_{x/a}\right|(2s) \quad (58)$$

$$\lim_{x'\to x} \partial_x \partial_{x'} \zeta_1(s, x\,|\,x') = \pi^{2-2s}a^{2s-3}[\zeta_R(2s-2) + \tfrac{1}{2}Z\left|{}^{0}_{x/a}\right|(2s-2)] \quad (59)$$

follow easily and for $\langle \hat{T}_{00} \rangle$ we find if (56), (57), (58) and (59) are substituted into (45)

$$\langle \hat{T}_{00} \rangle = -\lim_{s\to 1} (4\pi)^{-3/2}\pi^{5-2s}a^{2s-6}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}(2s-5)^{-1}\{[4\xi(s-1)+2-s]\zeta_R(2s-5)$$

$$+[1-6\xi+(2\xi-\tfrac{1}{2})(s-1)]Z\left|{}^{0}_{x/a}\right|(2s-5)\} \quad (60)$$

$$= -\frac{\pi^2}{12a^4}[\zeta_R(-3) + (1-6\xi)Z\left|{}^{0}_{x/a}\right|(-3)]. \quad (61)$$

For conformal coupling $\xi = \frac{1}{6}$ and (61) gives the standard answer. For minimal coupling $\xi = 0$ and (61) is equivalent to the expression quoted by DeWitt (1975), since

$$Z\left|{}^{0}_{x/a}\right|(-3) = \frac{3}{4\pi^4}Z\left|{}^{x/a}_{0}\right|(4),$$

from (52), and

$$\sum_{-\infty}^{\infty} (\theta - n)^{-4} = \pi^4(\operatorname{cosec}^4 \pi\theta - \tfrac{2}{3}\operatorname{cosec}^2 \pi\theta).$$

$Z\left|{}^{x/a}_{0}\right|(4)$ diverges as the plates are approached, $x \to 0$ or $x \to a$. The same result is obtained from the image method, throwing away the direct term.

Two questions now arise: What happens *on* the plates $x = 0$, $x = a$, and what is the total energy?

One point is that, in the definition of Z (equation (52)) if g is an integer the term $m = -g$ is excluded from the sum so that $Z\left|{}^{0}_{0}\right|(4)$ is *not* infinite. If one simply sets x equal to zero in (61) one finds the value

$$\langle \hat{T}_{00} \rangle_0 = \frac{\pi^2}{4a^4}(4\xi - 1)\zeta_R(-3) \quad (62)$$

the significance of which is not apparent.

Clearly if (61) is integrated over the slab, $0 \leq x \leq a$, the divergences will produce an infinite answer. Yet, if the value (62) is taken seriously, we must remember that at $x = 0$ and $x = a$, the relevant pole terms in the image sum $Z\left|{}^{x/a}_{0}\right|(4)$ are to be excluded from the integrand. This produces a finite total energy.

The same finite answer can be obtained if we integrate at an earlier stage. For example, the relation between the total energy E and the integrated zeta function can be used right from the start. For the slab the integrated zeta function is a Riemann zeta function,

$$\zeta_1(s) = (a/\pi)^{2s}\zeta_R(2s). \tag{63}$$

Then, from general theory, the total energy per unit plate area is

$$E = \frac{i}{2}\frac{\zeta_4(s-1)}{s-1}\bigg|_{s=1} = -\frac{1}{2}(4\pi)^{-3/2}\frac{\Gamma(s-\frac{5}{2})}{\Gamma(s)}\zeta_1(s-\tfrac{5}{2})|_{s=1} = -\frac{\pi^2}{12a^3}\zeta_R(-3), \tag{64}$$

which is the integral of the first term of (61), just as (63) is the integral of the first term of the coincidence limit of the local zeta function (58).

The total energy (64) is independent of ξ and this can be seen in another way. The correct expression for the operator $\vec{T}_{00}$ (see (25)) is

$$\vec{T}_{00}\Phi(x, x') = \tfrac{1}{2}i[\partial_0\,\partial_{0'} - \delta^{ij}\,\partial_i\,\partial_{j'} - 2\xi\delta^{ij}(\partial_i\,\partial_{j'} + \partial_i\,\partial_j)]\Phi(x, x'), \tag{65}$$

where $\Phi(x, x')$ is a biscalar. (To obtain (25), which is simpler computationally, the equations of motion have been employed. This is legitimate in the present case but is not to be recommended in general as it would lead to incorrect anomalies, for instance. The two forms agree when $\xi = 0$.)

Consider now the spatial integral of the coincidence limit of (65)

$$J = \int [\vec{T}_{00}(\xi)\Phi(x, x')]\,d^3x$$

and use Synge's relation

$$[\partial_j\,\partial_i\,\Phi] + [\partial_{j'}\,\partial_i\,\Phi] = \partial_j[\partial_i\,\Phi]$$

to write

$$J = \int_{M_3} [\vec{T}_{00}(0)\Phi]\,d^3x - i\xi\int_{\partial M_3} d\sigma^i[\partial_i\,\Phi] \tag{66}$$

(a similar result holds in curved space–time).

Equation (66) expresses a slight generalisation of the known fact (DeWitt 1975) that the improved stress–energy tensor differs from the minimal one by a divergence.

Classically Φ could be $\varphi(x)\varphi(x')$ and J would then be the total field energy. If the field satisfies D or N boundary conditions the surface term in (66) can be dropped and the total energy is independent of ξ.

This agrees with the construction of the quantum Hamiltonian $\hat{H}$,

$$\hat{H} = \tfrac{1}{2}\sum \omega_k(a_k a_k^\dagger + a_k^\dagger a_k)$$

where the ω_k are mode energies (Fulling 1973). These energies do depend on ξ in curved space but the point is that there is no extra, explicit ξ dependence.

The case we are interested in is when Φ is the Feynman Green function G so that J is the vacuum averaged energy E, $=\langle\hat{H}\rangle$, presumably. If Φ is taken to be some regularised form of $G(x, x')$, say $\zeta_4(s, x, x')$, the surface term in (66) goes out here as well, the argument being that the integrand in the second integral is evaluated strictly *on* ∂M_3 so that the x-dependent part of $\zeta_1(s, x, x)$ in (58) does not diverge on ∂M_3. In this case, then, J equals E and the value is given by (64). (Note that the actual construction of $\lim_{s\to 1}[\vec{T}_{00}\zeta_4(s, x, x')]$ gives the same answer whether one uses (25) or (65) for $\vec{T}_{00}$. The two forms differ by terms of the order of $\Box\zeta_4(1, x, x') = \zeta_4(0, x, x')$ which is zero here; or one can check explicitly.)

If one did retain all the pole terms and their divergences and if $\xi = \frac{1}{6}$, the boundary divergences in (66) would cancel those in the volume integration on the right-hand side to produce a finite answer, the left-hand side (DeWitt 1975). However it would be only for this value of ξ that the integral equals $\langle \hat{H} \rangle$. For any other value it would be infinite. This situation really refers to an integration not over the whole manifold but only over a region which approximates to it (to arbitrary accuracy). That is, for the slab, over the open region bounded by $x = \epsilon$ and $x = a - \epsilon$ with $\epsilon > 0$.

From the variational point of view the total energy can be defined from the change in the Lagrangian under the special variation $\delta g_{00} = 2\alpha\delta\alpha g_{00}$, with α constant. This variation does not vanish on the boundary, as variations are normally assumed to do, hence in addition to the integral of the local $\langle \hat{T}_{00} \rangle$ 'right up to the boundary', which will diverge, there will be, it is conjectured, an extra boundary contribution that cancels off the infinity.

6. The rectangular waveguide

This is discussed in Dowker and Kennedy (1978) and we present an additional exposition here. The new fact is that, even for the improved tensor, $\langle \hat{T}_{00} \rangle$ diverges as the corners of the rectangle are approached again rendering the naive total energy infinite. However the zeta function method produces a finite result for E which was explicitly evaluated in the above reference for two special rectangle shapes.

The integrated zeta function for the rectangle of sides a and b for D conditions is easily found to be

$$\zeta_2(s) = \tfrac{1}{2}\pi^{-2s}[\tfrac{1}{2}Z\left|\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right|(s, A^{-1}) - b^{2s}\zeta_R(2s) - a^{2s}\zeta_R(2s)] \tag{67}$$

using the same notation as in § 4.

Then the result $E = (1/8\pi)\zeta_2'(-1)$, together with the functional relations obeyed by Z and ζ_R, yield

$$E(a, b) = \frac{1}{32\pi}\left((a^{-2} + b^{-2})\zeta_R(3) - \frac{ab}{\pi} Z\left|\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right|(2, A)\right). \tag{68}$$

When the rectangle is a square and also when $b = 2a$ this result reduces to the ones given before (Dowker and Kennedy 1978).

The values of the function $E(r) = E(r^{1/2}, r^{-1/2})$, from which $E(a, b)$ can be found, are plotted in figure 2. It is 'symmetrical', $E(r) = E(r^{-1})$, about $r = 1$ (a square) where it has a maximum, and it passes through zero for $r \sim 1{\cdot}75$.

The local zeta function is also easily derived and can be written either in eigenfunction or in image form, analogously to (51), (53) and (55). We do not write them out. The image expression is probably best for an approximate numerical calculation.

A straightforward calculation based on the image form of the Green function and equation (25) yields the expression for the energy density, $\langle \hat{T}_{00} \rangle = T_{00}$:

$$T_{00}(x, y; b) = -\frac{1}{32\pi^2}\Bigg[\sum_{m,n=-\infty}^{\infty}{}' \left(\frac{1}{(b^2n^2 + m^2)^2} \pm \frac{4}{3}\frac{b^2n^2}{[b^2n^2 + (x-m)^2]^3} \pm \frac{4}{3}\frac{m^2}{[m^2 + b^2(\bar{y} - n)^2]^3}\right) + \frac{1}{3}\sum_{-\infty}^{\infty}\frac{1}{[(x-m)^2 + b^2(\bar{y} - n)^2]^2}\Bigg], \tag{69}$$

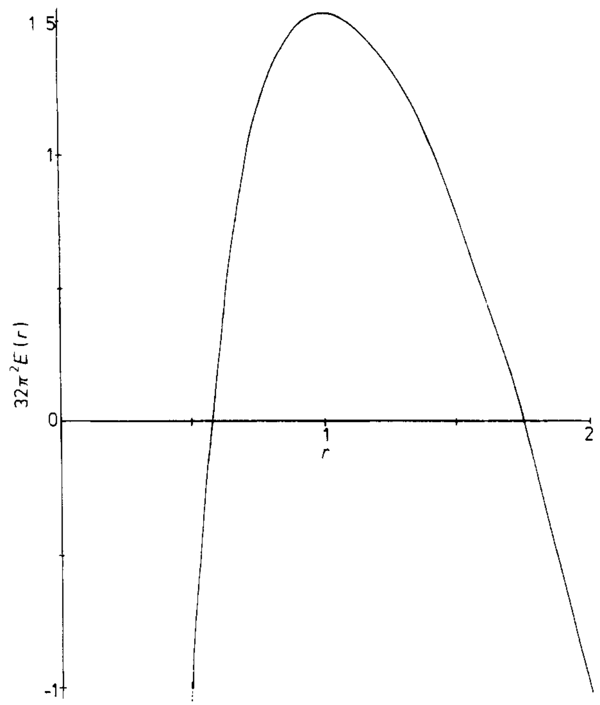

Figure 2. Total vacuum energy per unit z slice of a rectangular waveguide of unit area ($a = r^{1/2}$, $b = r^{-1/2}$) plotted as a function of r for Dirichlet boundary conditions.

($y = b\bar{y}$; $0 \leq \bar{y} \leq 1$). The + sign is for N and the − for D. We have chosen a rectangle with side $a = 1$.

The last sum in (69) is the one that contains the divergences as the corners are approached. The relevant terms are those with $(m, n) = (0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$, corresponding to the four corners. Close to a corner T_{00} goes like $-(1/96\pi^2 r^4)$ where r is the distance from the corner.

Numerical calculation shows that these four terms dominate the entire remainder of T_{00}, being at least a factor of 10 larger, at least for D conditions—the only case evaluated. For example, at the centre of a square the corner terms contribute an amount $-1{\cdot}333(1/8\pi^2)$ while the rest of the expression adds up to $0{\cdot}0884(1/8\pi^2)$. At the middle of a side the corresponding values are $-2{\cdot}77(1/8\pi^2)$ and $0{\cdot}142(1/8\pi^2)$.

The general conclusion is the same as in the slab case. When evaluating the total energy the zeta-function method effectively drops the corner poles, actually *at* the corner, so as to produce a finite answer. An infinite value is obtained if the local density (69) is integrated right up to the boundary.

7. 'Twisted' fields on the Klein bottle

These correspond to non-trivial representations, $a(\gamma)$, of the fundamental group of K_2. For the two-torus, T^2, the lattice group Γ is $Z_\infty \otimes Z_\infty$ and the $a(\gamma)$ would then be labelled by two real numbers α and β,

$$a(\gamma_{nm}) = \exp[2\pi i(n\alpha + m\beta)], \qquad 0 \leqslant \alpha,\ \beta \leqslant \tfrac{1}{2},$$

a simple extension of the one-dimensional expression (19).

By contrast the lattice group for K_2 is non-Abelian due to the reflection and so we require an Abelian representation of a non-Abelian group. This restricts the possibilities. We find the two sets of representations, p is an integer,

$$a(\gamma_{pm}) = \exp(2\pi i p\alpha),$$

and

$$a(\gamma_{pm}) = \exp(2\pi i p\alpha)(-1)^m, \qquad 0 \leqslant \alpha \leqslant \tfrac{1}{2}. \tag{70}$$

These mean that in the y-direction the field is either periodic or antiperiodic (changes sign). There is no such restriction in the x-direction unless, as in the circle case discussed in § 3, φ is real. Then there are four types of twisted scalar fields. A similar conclusion is reached by Hart and DeWitt (private communication).

In accordance with the general formulae (2), or (9), the phase factors (70) can be inserted into the image expression for $G(x, x')$, (26), or into that for $\langle \hat{T}_{00} \rangle$, (27). Equivalently, we can now write the partial coincidence limit of the 'twisted' zeta function on the Klein-bottle waveguide in terms of the Epstein zeta function,

$$\lim_{\substack{t' \to t \\ z' \to z}} \zeta_4(s, x, x'; \alpha, \beta)$$

$$= \frac{i}{16\pi^2} \frac{\Gamma(2-s)}{\Gamma(s)} \left(Z \left| \begin{matrix} \dfrac{x-x'}{2a} & \dfrac{y-y'}{2b} \\ 2\alpha & \beta \end{matrix} \right| (2-s, A) \right.$$

$$\left. + e^{2\pi i \alpha} Z \left| \begin{matrix} \dfrac{x-x'}{2a} + \dfrac{1}{2} & \dfrac{y+y'}{2b} \\ 2\alpha & \beta \end{matrix} \right| (2-s, A) \right) \tag{71}$$

where $0 \leqslant \alpha \leqslant \frac{1}{2}$ and β is either zero or one half.

As an example of the sort of changes that twisting can cause consider the energy density of a twisted field on the Möbius strip. Instead of (29) there is

$$\langle T_{00}^{(\alpha)} \rangle_{\mathcal{M}_2} = -\frac{1}{16\pi^2 a^4} \sum_{n=1}^{\infty} n^{-4} \cos(4\pi n\alpha) - \frac{4a^2}{3\pi^2} \sum_{n=0}^{\infty} \frac{(2n+1)^2 \cos[2\pi\alpha(2n+1)]}{[(2n+1)^2 a^2 + 4y^2]^3}.$$

If α is neither zero nor one half then we should really multiply this expression by two since φ must now be taken complex (we did this in § 3), however we shall ignore this point.

$\langle \hat{T}_{00}^{(\alpha)} \rangle_{\mathcal{M}_2}$ can be evaluated to give

$$\langle \hat{T}_{00}^{(\alpha)} \rangle_{\mathcal{M}_2} = \frac{\pi^2}{48a^4} \left[-\frac{1}{30} + 16\alpha^4 - 16\alpha^3 + 4\alpha^2 \right.$$

$$\left. + \left(\frac{\partial}{\partial \eta^2} \right)^2 \eta [\tanh \eta \cosh(4\alpha\eta) - \sinh(4\alpha\eta)] \right] \tag{72}$$

for any α. If $\alpha = \frac{1}{2}$ the calculation is trivial directly and is just the previous result, (29), with the last term reversed in sign. Then it is easily found that

$$\langle \hat{T}_{00}^{(1/2)} \rangle_{\mathcal{M}_2} \sim \begin{cases} (\pi^2/1440a^4)(19-24\eta^2) & \text{as } \eta \to 0 \\ (\pi^2/192a^4)(-\frac{2}{15}+\eta^{-3}) & \text{as } \eta \to \infty. \end{cases}$$

This distribution is also plotted in figure 1. The main effect of the twisting is to change the sign of the energy density near the centre of the Möbius band.

From (72) it is possible to determine the limiting forms for any α as

$$\langle \hat{T}_{00}^{(\alpha)} \rangle_{\mathcal{M}_2} \sim (\pi^2/480a^4)[-7+10(16\alpha^4-\tfrac{112}{3}\alpha^3+20\alpha^2)+8(1-20\alpha^2+80\alpha^4-64\alpha^5)\eta^2]$$

for $\eta \to 0$, and

$$\langle T_{00}^{(\alpha)} \rangle_{\mathcal{M}_2} \sim (\pi^2/480a^4)[-\tfrac{1}{3}+40(4\alpha^4-4\alpha^3+\alpha^2)+40\alpha^2\eta^{-1}\,\mathrm{e}^{-4\alpha\eta}] \tag{73}$$

for $\eta \to \infty$, if $0<\alpha<\frac{1}{4}$. If $\frac{1}{4} \leqslant \alpha < \frac{1}{2}$ then the last term in (73) reads $-40\bar{\alpha}^2\eta^{-1}\,\mathrm{e}^{-4\bar{\alpha}\eta}$ where $\bar{\alpha} = (\frac{1}{2}-\alpha)$.

For the intermediate value $\alpha = \frac{1}{4}$ the value of $\langle \hat{T}_{00}^{(1/4)} \rangle$ at $\eta = 0$ equals the limiting value as $\eta \to \infty$, namely $(7\pi^2/11520\alpha^4)$, in fact for this value of α, $\langle \hat{T}_{00} \rangle$ is constant. $\alpha = \frac{1}{4}$ corresponds to the field being antiperiodic in the x-direction on the double covering space of the Klein bottle/Möbius band.

The total energy for the Klein bottle can be found for the twisted fields. A straightforward calculation based on (71) yields the result,

$$E(a, b, \alpha) = -(ab/16\pi^2)Z\left|\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right|(2, A)$$

$$-(4/a^2)\sum_{n=1}^{\infty}[\cos(2\pi\alpha n) - \tfrac{1}{8}\cos(4\pi\alpha n)]n^{-3} \tag{74}$$

for $\beta = 0$, i.e. periodic fields in the y-direction. For antiperiodic fields ($\beta = \frac{1}{2}$), we find the total energy to be independent of α and equal to just the first term of (74).

The particular value $E(a, b, \frac{1}{2})$ is given as

$$E(a, b, \tfrac{1}{2}) = (7/16\pi^2)\zeta_{\mathrm{R}}(3) - (ab/16\pi^2)Z\left|\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right|(2, A)$$

which should be compared with (34), i.e. $E(a, b, 0)$. The first term has been reversed in sign.

The intermediate value $E(a, b, \frac{1}{4})$ is

$$E(a, b, \tfrac{1}{4}) = (5/64\pi^2)\zeta_{\mathrm{R}}(3) - (ab/16\pi^2)Z\left|\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right|(2, A).$$

Instead of plotting some of these expressions it is possible to use the values of the total energy $E(r^{1/2}, r^{-1/2})$ of a rectangular guide given in figure 2. From equation (68) we have that, for example,

$$E(r^{1/2}, r^{-1/2}, \tfrac{1}{2}) = 4E(r^{1/2}, r^{-1/2}) + \frac{1}{16\pi}\left(\frac{5}{r} - 2r\right)\zeta_{\mathrm{R}}(3).$$

8. Three-dimensional geometries

As mentioned in Dowker and Critchley (1976b), the spatial section M_3 of a flat space–time, $T \otimes M_3$, could be any of the Clifford–Klein space forms, R^3/Γ. The Klein bottle and rectangular waveguides are two, non-compact examples. All the possible types are displayed by Wolf (1967), the original work being done by Hantsche and

Wendt, for the compact cases. It is simply a question of using the point identifications that define these forms in order to write down the image form of the Green function and thence to find the averaged energy density.

For example another (orientable) non-compact variety is $\mathscr{K}_1$. This is defined in exactly the same way as the Klein-bottle waveguide, $\mathscr{K}_2$, except that for odd p the sign of z is reversed, i.e. we identify (x, y, z) and $(x+pa, (-1)^p y+2mb, (-1)^p z)$.

Now $\langle \hat{T}_{\mu\nu}\rangle$ becomes a function of z as well as of y and we find, e.g.,

$$\langle \hat{T}_{00}\rangle = -(1/32\pi^2) \sum_{n,m=-\infty}^{\infty}{}' (n^2a^2+m^2b^2)^{-2} - (1/6\pi^2) \sum_{n,m=-\infty}^{\infty} [s_{nm}^{-2} + 4a^2(2n+1)^2 s_{nm}^{-3}]$$

where

$$s_{nm} = (2n+1)^2a^2 + 4(y-mb)^2 + 4z^2.$$

In the limit $b\to\infty$, $\langle \hat{T}_{00}\rangle$ is symmetrical under rotations about the x-axis. Its profile is very similar to the variation of $\langle \hat{T}_{00}\rangle$ with y in the Möbius strip example (figure 1).

The other types of space-forms, 18 in all (Wolf 1967, p 123), can be treated similarly. However we baulk at writing down the results for each of them, essentially because of the lack of any physical motivation. (There is no technical difficulty.) Hart and DeWitt (private communication) have also given some explicit expressions for a few cases. Here we shall be contented to give the complete $\langle \hat{T}_{\mu\nu}\rangle$ for a space similar to the Klein-bottle waveguide, except that it is periodic in the z-direction (type $\mathscr{B}_1$ in Wolf 1967).

We find,

$$\langle \hat{T}_{\mu\nu}\rangle = -\frac{1}{4\pi^2} \sum_{\{l\}=-\infty}^{\infty} \Bigg[\eta_{\mu\nu} \bigg(\frac{(2-4\xi)A_{\mu l} + 2\xi(1+A_{\mu l}A_{\nu l})}{\sigma_l^4} + (4\xi-1)\frac{A_{\sigma l}\eta^{\sigma}{}_{\sigma}\sigma_l^2 + 4A_{\sigma l}z^{\sigma}{}_l z_{\sigma l}}{\sigma_l^6} \bigg) + 4[(2-4\xi)A_{\mu l} + 2\xi(1+A_{\mu l}A_{\nu l})]\frac{z_{\mu l}z_{\nu l}}{\sigma_l^6} \Bigg], \tag{75}$$

where

$$\boldsymbol{l} = (n, m, l)$$

$$\sigma_l^2 = n^2a^2 + \{2mb + [(-1)^n - 1]y\}^2 + l^2c^2$$

$$A_{\mu l} = \begin{cases} -1 & \text{if } \mu = 2 \text{ and } n \text{ odd} \\ 1 & \text{otherwise} \end{cases}$$

with

$$z_{\mu l}z_{\nu l} = 0 \qquad \text{if } \mu \text{ or } \nu = 0$$

and

$$\|z_{il}z_{jl}\| = \left\| \begin{array}{c|c|c} n^2a^2 & na\{2mb+[(-1)^n-1]y\} & nlac \\ na\{2mb+[(-1)^n-1]y\} & \{2mb+[(-1)^n-1]y\}^2 & lc\{2mb+[(-1)^n-1]y\} \\ nlac & lc\{2mb+[(-1)^n-1]y\} & l^2c^2 \end{array} \right\|.$$

c is the periodicity in the z-direction.

9. Curved-space examples

Another classic space-form problem is the 'spherical' one: find all three-spaces locally isometric to the three-sphere, S^3. The answer was given by Threlfall and Seifert (1930) and a modern discussion can be found in Wolf (1967). (See also Ellis (1971) for a useful summary of results.)

The corresponding space–times are $T\otimes S^3/\Gamma$ with Γ a discrete, freely acting subgroup, without fixed points, of the group of isometries of S^3, which is the product of left and right Clifford translations on S^3, $\mathrm{SU}(2)\otimes\mathrm{SU}(2)$, homomorphic to SO(4).

The analysis in § 2 spotlights the importance of quantum mechanics on S^3, which is a topic with a certain literature (Schulman 1968, Dowker 1970, 1971). The treatment is, or can be, purely group theoretical because S^3 is isometric to SU(2).

We shall use the formalism of § 2 to determine the effective Lagrangian $L^{(1)}$ for a scalar field on $T\otimes S^3/\Gamma$.

The eigenvalues and degeneracies for the conformally coupled scalar field on the Einstein universe, $T\otimes S^3$, are well known and are

$$\tilde{\lambda}_n=(n/a)^2,\qquad d_n=n^2,\qquad n=1,2,3\ldots$$

where a is the radius of S^3. Thus the integrated zeta function for S^3 is

$$\zeta_3(s)=\zeta_{S^3}(s)=\sum_{n=1}^{\infty}n^2(n^2/a^2)^{-s}=a^{2s}\zeta_{\mathrm{R}}(2s-2).$$

Then for the total energy we have, for real fields,

$$E=-L^{(1)}=\frac{\mathrm{i}}{2}\frac{1}{s-1}\frac{\mathrm{i}}{(4\pi)^{1/2}}\frac{\Gamma(s-\frac{3}{2})}{\Gamma(s-1)}\zeta_3(s-\tfrac{3}{2})|_{s=1}=\tfrac{1}{2}\zeta_3(-\tfrac{1}{2})=1/240a,\tag{76}$$

which is the standard result (Ford 1975, Dowker and Critchley 1976b). Note again that there are no divergences in the zeta-function approach since the a_2 coefficient is zero for the Einstein universe.

The density $\langle\hat{T}_{\mu\nu}\rangle$ can now be found by noting that $\langle\hat{T}^{\mu}{}_{\mu}\rangle$ is zero and that $\langle\hat{T}_{ij}\rangle$ must be proportional to g_{ij}. Further $\langle\hat{T}^0{}_0\rangle=E/2\pi^2a^3$ since space is homogeneous, and $\langle\hat{T}_{i0}\rangle=0$ since the space–time is static. The result is that given by Dowker and Critchley (1976b).

The simplest space covered by S^3 is $P^3=S^3/Z_2$, the projective three-sphere, obtained from S^3 by identifying antipodal points.

There are two, one-dimensional representations (reps) of Z_2 giving two sorts of scalar fields on P^3,

$$\left.\begin{aligned}a(1)&=1\\a(-1)&=1\end{aligned}\right\}\ \text{trivial rep A}\qquad\left.\begin{aligned}a(1)&=1\\a(-1)&=-1\end{aligned}\right\}\ \text{'twisted' rep B.}$$

The choice of the trivial representation kills all the half integral angular momentum representations in the expansions while the twisted one removes the integral angular momentum eigenvalues and functions (Schulman 1968, Dowker 1972a, b). Then for the zeta functions we find

$$\zeta_{\mathrm{A}}(s)=a^{2s}\sum_{j=0,1,\ldots}^{\infty}(2j+1)^{-2(s-1)}=a^{2s}\zeta_{\mathrm{R}}(2s-2)(1-2^{2-2s})$$

$$\zeta_{\mathrm{B}}(s)=a^{2s}\sum_{j=\frac{1}{2},\frac{3}{2},\ldots}^{\infty}(2j+1)^{-2(s-1)}=a^{2s}2^{2-2s}\zeta_{\mathrm{R}}(2s-2)$$

and for the total vacuum energies,

$$E_{\mathrm{A}} = -7/240a \tag{77}$$

$$E_{\mathrm{B}} = 1/30a. \tag{78}$$

The global group of isometries of P^3 is still spin(4) so that one would expect $\langle\hat{T}_{\mu\nu}\rangle$ to have the same tensor structure as that for S^3. Further, P^3 is homogeneous so that $\langle\hat{T}_{00}\rangle_{\mathrm{A,B}} = E_{\mathrm{A,B}}/\pi^2 a^3$ and thus the $\langle\hat{T}_{\mu\nu}\rangle$ for P^3 is just a numerical factor times that for S^3.

For the more general case, S^3/Γ, we can use the analysis of § 2. There are two basic classes of manifolds—those that are homogeneous and those that are not. We represent a point of S^3 by the group element, $q \in \mathrm{SU}(2)$. The global symmetry group of S^3 is then $\mathrm{SU}(2)_{\mathrm{L}} \otimes \mathrm{SU}(2)_{\mathrm{R}}$, exhibited as the action $q \to \xi q \eta$, where $\xi \in \mathrm{SU}(2)_{\mathrm{L}}$ and $\eta \in \mathrm{SU}(2)_{\mathrm{R}}$. The covering group Γ is likewise given as $\Gamma = \Gamma_{\mathrm{L}} \otimes \Gamma_{\mathrm{R}}$ where Γ_{L} and Γ_{R} are finite, freely acting subgroups of $\mathrm{SU}(2)_{\mathrm{L}}$ and $\mathrm{SU}(2)_{\mathrm{R}}$, without fixed points. Thus, S^3/Γ is obtained from S^3 by identifying the points q and $\gamma_{\mathrm{L}} q \gamma_{\mathrm{R}}$ where γ_{L} ranges over all elements of Γ_{L}, and likewise for γ_{R} and Γ_{R}. The homogeneous spaces are those for which Γ_{L} (or Γ_{R}) consists of just the unit element, **1**.

The zeta function on S^3/Γ can now be constructed from the eigenfunctions of the Laplace–Casimir operator or, from (10), using the fact that $\hat{\gamma}$ is a left translation times a right one. (The term 'translation' is not actually too indicative here since it is the fact that γ_{L} and γ_{R} belong to SU(2) groups, and therefore correspond to 'rotations', that we wish to bring out. The analysis is just angular momentum theory.)

For the non-local zeta function we find

$$\zeta_3(s, q', q'') = (2\pi^2 a^3)^{-1} a^{2s} \sum_{\gamma \equiv (\gamma_{\mathrm{L}}, \gamma_{\mathrm{R}})} a(\gamma) \sum_{l=1}^{\infty} l^{1-2s} \chi_l(q'\gamma_{\mathrm{R}} q''^{-1}\gamma_{\mathrm{L}}) \tag{79}$$

where $\chi_l(q)$ is the character of the l-representation of SU(2) on the element q, i.e.

$$\chi_l(q) = \sin(l\theta)/\sin\theta, \qquad (l = 2j+1),$$

where $a\theta$ is the radial distance on S^3 between the origin (= unit element) and the point q.

It is seen that if either Γ_{L} or Γ_{R} is trivial, $\zeta(s, q, q)$ is independent of q, which is a consequence of the homogeneity of the space.

Expression (79) can be written in terms of the one-dimensional Epstein zeta function, (52). Thus

$$\zeta_3(s, q', q'') = -(2\pi^2 a^3)^{-1} \tfrac{1}{2} a^{2s} \sum_{\gamma} a(\gamma)(\sin\theta_\gamma)^{-1} \frac{\partial}{\partial\theta_\gamma} Z\left|{}^{0}_{\theta_\gamma/2\pi}\right|(2s) \tag{80}$$

where $\theta_\gamma = \theta_\gamma(q', q'') = \theta(q'\gamma_{\mathrm{R}} q''^{-1}\gamma_{\mathrm{L}})$ is the 'radial angle' between the unit element and the point $q'\gamma_{\mathrm{R}} q''^{-1}\gamma_{\mathrm{L}}$.

We are now interested in finding the vacuum average of the stress–energy tensor $\langle\hat{T}_{\mu\nu}\rangle$ for these general spherical space-forms.

For the homogeneous case $\langle\hat{T}_{00}\rangle$ can be most easily found by first evaluating the total energy E, i.e. the integrated $\langle\hat{T}_0{}^0\rangle$, and then dividing by the volume since $\langle\hat{T}_0{}^0\rangle$ will be independent of position. Also E is the negative of the effective Lagrangian so we might as well obtain this quantity before going on to the local densities. Firstly (80) is written in terms of 'images' by virtue of (54) as

$$\zeta_3(s, q', q'') = -\tfrac{1}{2}(2\pi^2 a^3)^{-1} a^{2s} \pi^{2s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}-s)}{\Gamma(s)} \sum_\gamma a(\gamma)(\sin\theta_\gamma)^{-1} \frac{\partial}{\partial\theta_\gamma} Z\big|^{\theta_\gamma/2\pi}_{0}\big|(1-2s). \quad (81)$$

Then (compare (76)), the Lagrangian density is given by

$$\begin{aligned}\mathscr{L}^{(1)}(q) &= -\tfrac{1}{2}\zeta_3(-\tfrac{1}{2}, q, q)\\ &= -(2\pi^2 a^4)^{-1}\left(\frac{1}{240} + \frac{1}{8\pi^2}\sum_{\gamma\neq\mathbf{1}} a(\gamma)(\sin\theta_\gamma)^{-1}\frac{\partial}{\partial\theta_\gamma} Z\big|^{\theta_\gamma/2\pi}_{0}\big|(2)\right) \qquad (82)\end{aligned}$$

where we have separated off the term with $\gamma_L = \mathbf{1}$, $\gamma_R = \mathbf{1}$ (i.e. $\gamma = \mathbf{1}$). In (82), $\theta_\gamma = \theta(q\gamma_L q^{-1}\gamma_R)$ and is independent of q if either γ_L or γ_R equals the unit element.

The integrated zeta function is already given quite generally in § 1 by equation (12) but of course also follows from (81) by setting q' and q'' equal and then integrating over S^3/Γ. If the functional relation,

$$\int_{SU(2)} \chi_l(q\gamma_L q^{-1}\gamma_R)\,\mathrm{d}q = (2\pi^2 a^3) l^{-1} \chi_l(\gamma_L)\chi_l(\gamma_R),$$

is employed we have

$$\zeta_3(s) = |\Gamma|^{-1} a^{2s} \sum_\gamma a(\gamma) \sum_{l=1}^{\infty} l^{-2s} \chi_l(\gamma_L)\chi_l(\gamma_R) \qquad (83)$$

which is again expressible as an Epstein zeta function. Thus we find

$$\zeta_3(s) = |\Gamma|^{-1} a^{2s} \sum_\gamma a(\gamma)(4\sin\theta_L \sin\theta_R)^{-1}[Z|_{(\theta_L-\theta_R)/2\pi}^{0}|(2s) - Z|_{(\theta_L+\theta_R)/2\pi}^{0}|(2s)]$$

where θ_L, $= \theta(\gamma_L)$, and θ_R, $= \theta(\gamma_R)$, depend on γ.

The image form of this expression is, from (54),

$$\begin{aligned}\zeta_3(s) = |\Gamma|^{-1} a^{2s} \pi^{2s-\frac{1}{2}} \frac{\Gamma(\frac{1}{2}-s)}{\Gamma(s)} \sum_\gamma a(\gamma)\\ \times (4\sin\theta_L\sin\theta_R)^{-1}[Z|^{(\theta_L-\theta_R)/2\pi}_{0}|(1-2s) - Z|^{(\theta_L+\theta_R)/2\pi}_{0}|(1-2s)].\end{aligned}$$

The evaluation of the local densities $\langle\hat{T}_{\mu\nu}\rangle$ is more complicated. We write them as coincidence limits, as in (45),

$$\langle\hat{T}_{\mu\nu}\rangle = 2(\mathrm{Re}\ \vec{T}_{\mu\nu}\zeta_4(1, t, q, t', q')).$$

The factor 2Re is inserted when it is required to consider complex fields as, e.g., when $a(\gamma)$ is not a real representation. It will not be explicitly written into the following equations. (This rather cavalier treatment of complex fields can be made more precise if the charge matrix formalism is used.)

The expressions for $\vec{T}_{\mu\nu}$ can be found in DeWitt (1975), for example, but will be repeated here for convenience. Thus

$$\vec{T}_{00} = \mathrm{i}[\partial_0\partial_0 + (2\xi - \tfrac{1}{2})(\partial_0\partial_0 - \nabla^{i}\nabla_{i'}) + \xi(\Box^2 + \Box'^2) + \tfrac{1}{2}\xi R]$$

and

$$\begin{aligned}\vec{T}_{ij} = \mathrm{i}[(2\xi-1)\nabla_i\nabla_{j'} + g_{ij'}(2\xi - \tfrac{1}{2})(\partial_0\partial_0 - \nabla^{i}\nabla_{i'})\\ -\xi(\nabla_{i'}\nabla_{j'} + \nabla_i\nabla_j) + \xi g_{ij'}(\Box^2 + \Box'^2) + \tfrac{1}{6}\xi R g_{ij'}].\end{aligned}$$

From (79) it is easily proved that $\zeta_3(s, q', q'') = \zeta_3^*(s, q'', q')$, so that $\langle \hat{T}_{ij} \rangle$ will be symmetrical in i and j, as required. If $a(\gamma)$ is real, i.e. $a(\gamma) = a(\gamma^{-1})$, then $\zeta_3(s, q, q')$ is symmetrical in q and q'. Otherwise, because χ_l is real the effect of taking the real part is to have $\frac{1}{2}(a(\gamma) + a(\gamma^{-1}))$ in place of $a(\gamma)$.

Although in this section we are interested only in the value $\xi = \frac{1}{6}$, the expressions for $\hat{T}_{\mu\nu}$ have been written down in generality, for future reference.

It is convenient to project $\langle \hat{T}_{\mu\nu} \rangle$ onto the Killing vectors of the local isometry group of $T \otimes S^3/\Gamma$, which is $E^1 \otimes \mathrm{SU}(2) \otimes \mathrm{SU}(2)$. $\langle \hat{T}_{00} \rangle$ remains the same while $\langle \hat{T}_{ij} \rangle$ is replaced by

$$\langle \hat{T}_{ab} \rangle = A^i_a A^j_b \langle \hat{T}_{ij} \rangle, \qquad a, b, = 1, 2, 3,$$

where the $A^i_a(q)$ are the Killing vector fields of the left SU(2) group. The right ones could have been chosen, or some combination of the two.

The idea, in the technical evaluation of $\langle \hat{T}_{\mu\nu} \rangle$, is to reduce the problem to one in angular momentum theory. A start is made by replacing the derivatives ∇_i by the generators, X_a, of the left group,

$$X_a = A^i_a(q)\,\partial_i$$

satisfying

$$X_a X_b - X_b X_a = c_{ab}{}^c X_c, \qquad \text{with } c_{ab}{}^c = 2\epsilon_{abc}/a.$$

Then, firstly,

$$\langle \hat{T}_{00} \rangle = \mathrm{i}[\![\{\partial_0\,\partial_0 + (2\xi - \tfrac{1}{2})[(\boldsymbol{X}^2 + \boldsymbol{X}'^2)/2 + \delta^{ab} X_a X'_b]\}\zeta_4(1, t, q, t', q')]\!]$$

where $\boldsymbol{X}^2$ $(= \delta^{ab} X_a X_b)$ is the Laplacian on S^3, and the Casimir operator on SU(2), while

$$\langle \hat{T}_{ab} \rangle = \mathrm{i}[\![\{(X_{(a}X_{b)} + X'_{(a}X'_{b)})/2 + (2\xi - 1)[(X_{(a}X_{b)} + X'_{(a}X'_{b)})/2 + X_a X'_b]$$
$$- \delta_{ab}(2\xi - \tfrac{1}{2})[(\boldsymbol{X}^2 + \boldsymbol{X}'^2)/2 + \boldsymbol{X}\,.\,\boldsymbol{X}'] - \delta_{ab} 2\xi/a^2\}\zeta_4(1, t, q, t, q')]\!].$$

Some slight manipulations have been made to get these forms, the point of them being that when $\boldsymbol{X}'$, acting on the Green function, is equivalent to $-\boldsymbol{X}$, the terms in the square brackets vanish. It can also be seen that, in general, $\boldsymbol{X}'^2$ can be replaced by $\boldsymbol{X}^2$ in the coincidence limit.

The next step is to eliminate the space–time zeta function in favour of ζ_3 so that expression (79) can be used. (46) yields,

$$\langle \hat{T}_{00} \rangle = \tfrac{1}{2}\zeta_3(-\tfrac{1}{2}, q, q) - \tfrac{1}{2}(2\xi - \tfrac{1}{2})[\![[\boldsymbol{X}^2 + \boldsymbol{X}\,.\,\boldsymbol{X}']\zeta_3(\tfrac{1}{2}, q, q')]\!] \tag{84}$$

$$\langle \hat{T}_{ab} \rangle = -\tfrac{1}{2}[\![\{(X_{(a}X_{b)} + X'_{(a}X'_{b)})/2 + (2\xi - 1)[(X_{(a}X_{b)} + X'_{(a}X'_{b)})/2 + X_a X'_b]$$
$$- \delta_{ab}(2\xi - \tfrac{1}{2})[\boldsymbol{X}^2 + \boldsymbol{X}\,.\,\boldsymbol{X}'] - \delta_{ab} 2\xi/a^2\}\zeta_3(+\tfrac{1}{2}, q, q')]\!]. \tag{85}$$

Equation (79) displays ζ_3 as a sum of terms involving the character $\chi_l(q\gamma_{\mathrm{R}} q'^{-1}\gamma_{\mathrm{L}})$. Because the sums over γ_{L} and γ_{R} can be re-ordered, this is equivalently written (in the sum) as

$$\chi_l(q\gamma_{\mathrm{R}} q'^{-1}\gamma_{\mathrm{L}}) = \chi_l(q'\gamma_{\mathrm{R}}^{-1} q^{-1}\gamma_{\mathrm{L}}^{-1}) \equiv \chi_l(q'\gamma_{\mathrm{R}} q^{-1}\gamma_{\mathrm{L}}) \tag{86}$$

where we have used the facts that $a(\gamma^{-1}) = a^*(\gamma)$ and $\chi_l(q^{-1}) = \chi_l(q) = \chi_l(gqg^{-1})$.

The effect of the generators $\boldsymbol{X}$ and $\boldsymbol{X}'$ on the character is required. This is a standard question, but perhaps the details will be useful. Firstly consider the quantity

$X_a\chi(q\gamma_R q'^{-1}\gamma_L)$ and, for convenience, let us denote the point $q\gamma_R q'^{-1}\gamma_L$ by q''. Now the equation $q''=q\gamma_R q'^{-1}\gamma_L$ is a right motion taking q into q'' and, since the Lie derivative of the Killing vector fields A^i_a is zero for right transformations, it is easy to show that $\boldsymbol{X}$ is equivalent to $\boldsymbol{X}''$. A similar result holds for $\boldsymbol{X}'$, if we use the alternative argument of χ given in (86), namely $q'''=q'\gamma_R q^{-1}\gamma_L$.

Inspection of (84), (85) shows then that one of the things needed is $X_aX_b\chi(q)$, where q is, for the moment, a generic point (actually either q''' or q''). Quite generally we must have

$$X_aX_b\chi_l(q)=A_l(q)\delta_{ab}+B_l\hat{q}_a\hat{q}_b+C_l(q)c_{ab}{}^c\hat{q}_c \tag{87}$$

where the A, B and C are functions of the angle $\theta(q)$, just like $\chi(q)$, and $\hat{\boldsymbol{q}}$ is the unit vector at the origin tangent to the geodesic connecting q to the origin.

Use of the commutation rules for the $\boldsymbol{X}$ and the easily verified result

$$X_a\chi_l(q)=(1/a)\hat{q}_a\chi'_l(q) \qquad \text{with } \chi'_l(q)\equiv\partial\chi_l(q)/\partial\theta$$

enables us to determine C as

$$C_l=(1/2a)\chi'_l(q).$$

The evaluation of A and B is a little more involved, but not much. The trace of (87) produces

$$3A_l+B_l=\boldsymbol{X}^2\chi_l=[(1-l^2)/a^2]\chi_l \tag{88}$$

since the characters are eigenfunctions of the Casimir operator. Next, (87) yields

$$A_l+B_l=\hat{q}^a\hat{q}^bX_aX_b\chi_l(q)=-(4/a^2)\hat{q}^a\hat{q}^b\ \mathrm{tr}(J_bJ_aD(q))=-(4/a^2)\,\mathrm{tr}((\boldsymbol{J}\cdot\hat{\boldsymbol{q}})^2D(q))$$

where $D(q)$ is the SU(2) representation matrix in the representation for which $\boldsymbol{J}$ is the generator, $\boldsymbol{J}^2=(l^2-1)/4$, i.e.

$$D(q)=\exp(\mathrm{i}2\theta\hat{\boldsymbol{q}}\cdot\boldsymbol{J})$$

for our normalisation. Thus

$$A_l+B_l=a^{-2}\frac{\partial^2}{\partial\theta^2}\,\mathrm{tr}\,D=a^{-2}\frac{\partial^2}{\partial\theta^2}\chi_l(q), \tag{89}$$

and A and B are found.

Expression (87) enables us to find the X_aX_b and $X'_aX'_b$ terms in (84) (85). In fact since $q'''=q''$ in the coincidence limit, $q=q'$, the $\boldsymbol{X}'\boldsymbol{X}'$ terms contribute exactly the same as the $\boldsymbol{XX}$ ones, in this limit.

The mixed term, $X_aX'_b$, is more trouble. One approach is to change the X'_b into an X_c and then use (87). From the fact that the Lie derivative of the A^i_a under a left transformation is given by

$$\pounds_aA^i_b=c_{ab}{}^c\,A^i_c$$

it is straightforward to show that

$$X'_b\chi_l(q'')=-D^c{}_b(\gamma_L^{-1})X''_c\chi_l(q''), \qquad q''=q\gamma_R q'^{-1}\gamma_L,$$

where $D^c_b(g)$ is the spin-one ('adjoint') representation matrix corresponding to the group element g.

Then

$$X_a X'_b \chi_l(q'') = -D^c{}_b(\gamma_L^{-1}) X''_a X''_c \chi_l(q''). \tag{90}$$

Similarly we have

$$X_b X'_a \chi_l(q''') = -D^c{}_b(\gamma_L^{-1}) X'''_a X'''_c \chi_l(q''')$$

so that, because $q''' = q''$ in the coincidence limit, $\langle \hat{T}_{ab} \rangle$ is symmetrical.

Another way of showing this necessary symmetry is directly from (79), (85), (87) and (90). Although not strictly required, it is instructive to go through this alternative method. From (79), (90) and (87) it is seen that we must consider the expression, coming from the $X_a X'_b$ term in (85),

$$-\sum_\gamma [D^c{}_b(\gamma_L^{-1}) A(q'')\delta_{ac} + \hat{q}''_a \hat{q}''_c D^c{}_b(\gamma_L^{-1}) B(q'') + D^c{}_b(\gamma_L^{-1}) c_{ac}{}^d \hat{q}''_d C(q'')] \operatorname{Re} a(\gamma) \tag{91}$$

with $q'' = q\gamma_R q^{-1}\gamma_L$.

The basic idea is to use the fact that A, B, and C are unchanged if q'' is replaced by its transformed inverse, $\bar{q}'' = q\gamma_R^{-1} q^{-1}\gamma_L^{-1}$ $(= \gamma_L q''^{-1}\gamma_L^{-1})$ in order to re-arrange the sum over γ by setting $\gamma \to \gamma^{-1}$ (i.e. $\gamma_L \to \gamma_L^{-1}$ and $\gamma_R \to \gamma_R^{-1}$). This combined operation restores the arguments of A, B and C to q'', in (91), but changes that of the D to γ_L and replaces $\hat{q}''$ by $\hat{\bar{q}}''$. The relation between q''^{-1} and $\bar{q}''$ is an adjoint transformation so that

$$-q''^b = D^b{}_a(\gamma_L^{-1})\bar{q}''^a$$

is the connection between the two sets of canonical parameters ($q''^a = q''_a$).

A property of the adjoint representation matrix D that is needed is that it is real, which implies that its inverse and transpose are the same. Further, we have the generator property

$$-c^d_{ac} D^c{}_b(\gamma) = D^d{}_c(\gamma) c^c_{be} D^e{}_a(\gamma^{-1}).$$

With these relations it is straightforward to check that interchanging a and b in (91) is equivalent to the above combined operation, under which the sum is unchanged, so that (91) is symmetric.

Considerable simplifications occur for the homogeneous case, i.e. when either Γ_L or Γ_R is the trivial group. Thus, when Γ_R is composed of just the unit element, it is easy to show from (90), or from (91), using

$$X_a D^b{}_c(q) = -c^b_{ad} D^d{}_c(q),$$

that, in the limit $q' \to q$ (when $q'' = \gamma_R$),

$$X_a X'_b \chi_l(q'') = -X''_b X''_a \chi_l(q'').$$

Applied to equations (84) and (85), this formula, or rather its symmetrical part, shows that the terms in square brackets vanish and we have the comparatively simple expressions

$$\langle \hat{T}_{00} \rangle = \tfrac{1}{2}\zeta_3(-\tfrac{1}{2}, q, q) \tag{92}$$

$$\langle \hat{T}_{ab} \rangle = -\tfrac{1}{6}\delta_{ab}[a^{-2}(1-6\xi)\zeta_3(\tfrac{1}{2}, q, q) + \zeta_3(-\tfrac{1}{2}, q, q)]$$

$$-\tfrac{1}{2}(2\pi^2 a^3)^{-1} a \sum_\gamma a(\gamma)(\hat{\gamma}_a\hat{\gamma}_b - \tfrac{1}{3}\delta_{ab}) \sum_l B_l(\gamma) l^{1-2s}\Big|_{s=1/2}. \tag{93}$$

Here $\zeta_3(s, q, q)$ is given by equation (79) and B_l follows from (88) and (89) as

$$B_l(\gamma)=\frac{1}{2a^2}\left(3\frac{\partial^2}{\partial\theta^2}+l^2-1\right)\chi_l(\gamma). \tag{94}$$

We have also set $\gamma_L=\gamma$.

As a check we notice that when $\xi=\frac{1}{6}$, $\langle\hat{T}^\mu{}_\nu\rangle$ is traceless, as required. Further, equation (92) agrees with the general result, mentioned earlier, that in the homogeneous case the energy density can be found from the total energy by dividing by the volume. Then $\langle\hat{T}_{00}\rangle$ should be the negative of the Lagrangian density. A comparison of (82) with (92) shows that this is so.

In the cases $\Gamma_L=\mathbf{1}$, and $\Gamma_L=Z_2$ we expect the terms proportional to $(\hat{\gamma}_a\hat{\gamma}_b-\frac{1}{3}\delta_{ab})$ to vanish from general symmetry considerations and explicit calculation confirms this. In the other cases the set of image points (of the origin, say), $\{\gamma\}$ provides a set of intrinsic directions—those of the geodesics connecting the images to the origin—so that the geometrical structure of $\langle\hat{T}_{ab}\rangle$ is correspondingly richer.

Specific forms for (82) and (93) will now be given. The detailed algebra is not particularly interesting. Again we can use either the eigenfunction form, (80), or the classical path form, (81). Repeated application of the formula

$$2\sum_{l=1}^{\infty}\sin(l\theta)=\sum_{m=-\infty}^{\infty}(m\pi+\tfrac{1}{2}\theta)^{-1}=\cot(\theta/2)$$

yields the results

$$\mathscr{L}^{(1)}(q)=-\frac{1}{2\pi^2a^4}\left(\frac{1}{240}-\frac{1}{16}\sum_{\gamma\neq\mathbf{1}}a(\gamma)\operatorname{cosec}^4(\tfrac{1}{2}\theta_\gamma)\right) \tag{95}$$

and

$$\langle\hat{T}_{ab}\rangle=-\tfrac{1}{6}\delta_{ab}\zeta_3(-\tfrac{1}{2},q,q)$$
$$-\frac{1}{8\pi^2a^4}\sum_{\gamma\neq\mathbf{1}}a(\gamma)(\hat{\gamma}_a\hat{\gamma}_b-\tfrac{1}{3}\delta_{ab})[\operatorname{cosec}^2(\tfrac{1}{2}\theta_\gamma)-\operatorname{cosec}^4(\tfrac{1}{2}\theta_\gamma)]. \tag{96}$$

Equation (95) is valid in all cases and, in the homogeneous case, also equals the negative of $\langle\hat{T}_{00}\rangle$, (92). The total energy is given generally by,

$$E=\frac{|\Gamma|^{-1}}{a}\left[\frac{1}{240}-\frac{1}{16}\sum_{\gamma\neq\mathbf{1}}a(\gamma)\operatorname{cosec}^2\left(\frac{\theta_L-\theta_R}{2}\right)\operatorname{cosec}^2\left(\frac{\theta_L+\theta_R}{2}\right)\right]. \tag{97}$$

We have now set ξ equal to $\frac{1}{6}$. For general ξ, (92) and (93) are still true except that the l^{-2s} factor in (93) and in the definition of ζ_3, (79), is to be replaced by $(l^2+6\xi-1)^{-s}$. The effects of this will not be given here.

As a simple example it can be checked that (95) and (96) reproduce the previous results for $P^3=S^3/Z_2$, equations (77), (78). For this case there is only one θ_γ for $\gamma\neq\mathbf{1}$, i.e. π. P^3 is a particular example of the 'lens-space', S^3/Z_m, for which there are $(m-1)$ θ_γ's, for $\gamma\neq\mathbf{1}$, i.e. $\theta_n=n(2\pi/m)$ for $n=1,2,\ldots,m-1$.

The representations $a(\gamma)$ of Z_m are generated by the mth roots of unity, $1,\omega,\omega^2,\ldots,\omega^{m-1}$. For example, the trivial representation is generated by 1 and the twisted (real) one by $\omega^{m/2}=-1$, if m is even. If m is odd there are no twisted real fields. $(H^1(S^3/\Gamma;Z_2)\sim H^1(\Gamma;Z_2)\sim 0$ if $\Gamma=Z_m$, m odd.)

$\langle\hat{T}_{\mu\nu}\rangle$ can be calculated for the lens-spaces using (95) and (96) on substitution of the specific values of $\theta_\gamma=\theta_n$ and $\operatorname{Re}a(\gamma)=\operatorname{Re}a(n)=\cos(2r\pi n/m)$, where ω^r is the generator of the representation $a(\gamma)$, $r = 0, 1, 2, \ldots, m-1$. Also, in accordance with some previous remarks, we have taken the real part of $a(\gamma)$, so that $\langle \hat{T}_{\mu\nu} \rangle$ must be multiplied by two for genuinely complex fields.

For homogeneous lens-spaces $\hat{\gamma}_a \hat{\gamma}_b$ is the same for all pre-images (except the antipode) and so the factor $(\hat{\gamma}_a \hat{\gamma}_b - \frac{1}{3}\delta_{ab})$ in (96) can be taken outside the summation. (Then, if desired, one of the *dreibeine* axes at the origin can be taken parallel to one $\hat{\gamma}$.)

The sums can be done if $r = 0$ (the untwisted case) to give the simple formulae

$$\langle \hat{T}_{00} \rangle = -(1440\pi^2 a^4)^{-1}(m^4 + 10m^2 - 14)$$

$$E = -(720a)^{-1}(m^3 + 10m - 14m^{-1})$$

$$\langle \hat{T}_{ab} \rangle = (360\pi^2 a^4)^{-1}[\tfrac{1}{12}\delta_{ab}(m^4 + 10m^2 - 14) + (\hat{\gamma}_a \hat{\gamma}_b - \tfrac{1}{3}\delta_{ab})(m^2 - 4)(m^2 - 1)]. \qquad (98)$$

Clearly, the corresponding expressions for other values of r and any specific m are easy to evaluate. As a sample the total energies on the space S^3/Z_4 for the twisted and untwisted cases are found to be

$$E = \begin{cases} -67/480a, & \text{untwisted} \\ +53/480a, & \text{twisted real.} \end{cases}$$

Since the physical relevance of these calculations is somewhat obscure it is not called for now to give more examples of these lens-spaces. However, for variety, we might be permitted to consider an example of a 'prism-space', S^3/D'_m, where D'_m is the binary, or double, dihedral group of order $2m$ (e.g. Hamermesh 1962, Coxeter 1974). When $m = 2$, D'_2 is the eight-group or quaternion-group. The seven angles θ_γ, $\gamma \neq \mathbf{1}$, are π, a group of three $\frac{1}{2}\pi$'s and a group of three $\frac{3}{2}\pi$'s. For the total energy there results,

$$E = \begin{cases} -187/960a & \text{untwisted} \\ 53/960a & \text{twisted real.} \end{cases}$$

Interesting geometrical and topological information about these, and the other spaces, can be found in the classic article by Threlfall and Seifert (1930).

Simply for the record we present the values of the total energy for the remaining homogeneous space-forms:

Octohedron-space, (S^3/T'),

$$E = -3761/8640a, \text{ (trivial). (No twisted real fields.)}$$

Truncated-cube-space, (S^3/O'),

$$E = -11321/17280a, \text{ (trivial); } = 3799/17280a, \text{ (twisted).}$$

Dodecahedron-space, (S^3/Y'),

$$E = -43553/43200a, \text{ (trivial). (No twisted real fields.)}$$

These numbers would appear to have only a curiosity value and so we do not give the full $\langle \hat{T}_{\mu\nu} \rangle$. The non-homogeneous cases are also passed over at this time except to say that the various quantities $\mathscr{L}^{(1)}(q)$, $\langle \hat{T}_{\mu\nu} \rangle(q)$ etc will be constant on the orbits of the global isometry group (which is the centraliser of Γ in $SU(2) \otimes SU(2)$). This is also clear from the actual construction of these quantities, which depend only on the element $q\gamma_R q^{-1}\gamma_L$.

There is no problem in extending the calculations to the hyperbolic spaces, H^3/Γ, although the complete classification of these is not known.

10. Comments and conclusion

The specific results we have obtained for $\langle\hat{T}_{\mu\nu}\rangle$ are probably somewhat academic but they are, we feel, amusing combinations of various interesting pieces of analysis. One hope is that the general formalism and particular calculational methods will prove of value.

The analysis can be extended to vector-valued fields, e.g. spinors. The methods will be detailed at another time. Although the Klein bottle and Möbius band are not parallelisable it is possible to patch up a twisted spinor theory by using the freedom provided by the $a(\gamma)$. The resulting $\langle\hat{T}_{\mu\nu}\rangle$ are somewhat complicated.

The finite temperature versions of our results are straightforwardly found and, again, will be reported in a later communication.

It is possible to use the vacuum-averaged $\langle\hat{T}_{\mu\nu}\rangle$ on the right-hand side of Einstein's field equations in the hope that this represents, in some way, the back-reaction of the field on the geometry. From the forms of $\langle\hat{T}_{\mu\nu}\rangle$ that we have obtained it is clear that twisted fields have a very different back-reaction to untwisted ones, as already remarked by Isham (1978). Equations (23), (24) and (77), (78) show the typical sign change. It is also to be noted, from say (76), (77) and (78), that altering the topology can change the sign of the energy. It is the twisted field on P^3 that has the same sign of the energy as the field on S^3, and this seems to be true for the other space-forms as well. Thus, if one asks for self-consistency via Einstein's equations, it is only the vacuum fluctuations of a twisted field that can support an Einstein universe of elliptic (P^3) spatial topology. However a difficulty arises if self-consistency is required for the other spherical space-forms. The general expression for $\langle\hat{T}_{ab}\rangle$, (96), is not of the same form as $R_{ab}-\frac{1}{2}g_{ab}R$ since it contains additional geometrical structure.

A number of points require elaboration. Fundamentally one needs a better understanding of the connection between the field theory and the Schwinger–DeWitt 'quantum mechanics' so that the significance of the $a(\gamma)$ factors can be further considered.

Finally we would like to thank C F Hart for sending a preliminary version of a paper concerned with the vacuum-averaged stress–energy tensor on the Euclidean space-forms. Where our results overlap they agree.

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